TA homework9 trans

Problem 1 — Virial Theorem

(a) Deriving the Ideal Gas Equation from the Virial Theorem

Since there are no interparticle interactions in an ideal gas, the right-hand side consists entirely of the gas's interaction with the container walls.

Pressure on the wall:

$$\mathbf{F} = P d\mathbf{A}$$

Integrating over the entire wall area:

$$\sum_i \langle \mathbf{r}_i \cdot \mathbf{F}_i \rangle = \oint P\mathbf{r} \cdot d\mathbf{A}$$

For any closed volume:

$$\oint \mathbf{r} \cdot d\mathbf{A} = 3V$$

Therefore:

$$\sum_i \langle \mathbf{r}_i \cdot \mathbf{F}_i \rangle = 3PV$$

Left side:

$$\langle K \rangle = \frac{3}{2} NkT$$

Substituting into Virial Theorem:

$$2\left(\frac{3}{2}NkT\right) = 3PV \implies \boxed{PV = NkT}$$

This is the ideal gas law.

(b) Virial under Gravity: Find the total mass of the galaxy cluster

For gravitational potential:

$$U = -\frac{GMm}{r}$$

For gravitational potential, the following holds:

$$\mathbf{r} \cdot \mathbf{F} = -U$$

Virial Theorem:

$$2K + U = 0$$

Kinetic energy:

$$K = \frac{1}{2} M \langle v^2 \rangle$$

Gravitational potential energy (for spherical mass distribution):

$$U \approx -\frac{GM^2}{R}$$

Substituting:

$$2\left(\frac{1}{2}M\left\langle v^2\right\rangle\right) - \frac{GM^2}{R} = 0 \implies \boxed{M = \frac{R\left\langle v^2\right\rangle}{G}}$$

Problem 2

Gaseous gas with two variables a_1, a_2, energy:

$$E = \frac{1}{2}(a_1^2 + a_2^2)$$

(a) Prove the transformation is a rotation matrix transformation:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Requires energy to remain unchanged:

$$b_1^2 + b_2^2 = a_1^2 + a_2^2$$

Written in matrix form:

$$(a_1, a_2)\begin{pmatrix} A^2 + C^2 & AB + CD \\ AB + CD & B^2 + D^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (a_1^2 + a_2^2)$$

For any a_1, a_2, the following must be satisfied:

$$A^2 + C^2 = 1, \quad B^2 + D^2 = 1, \quad AB + CD = 0$$

This is the condition for a two-dimensional orthogonal matrix.
If $\det > 0$, then the matrix is a rotation, not a reflection:

$$A = D = \cos\theta, \quad B = -\sin\theta, \quad C = \sin\theta$$

Therefore:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Collision of two particles: From the definition of momentum + relative momentum of $\mathbf{p}_1, \mathbf{p}_2 \to$ **CM**:

$$P = p_1 + p_2$$
$$p' = \mu\left(\frac{p_1}{m_1} - \frac{p_2}{m_2}\right), \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

The transformation matrix can be written as an angle θ, satisfying:

$$\tan\theta = \sqrt{\frac{m_2}{m_1}}$$

(b) If for any θ we have $\langle b_1 b_2\rangle = 0$, then the equipartition theorem must be satisfied.

Calculate:

$$b_1 = a_1\cos\theta - a_2\sin\theta$$
$$b_2 = a_1\sin\theta + a_2\cos\theta$$

Calculate the average:

$$\begin{aligned}\langle b_1 b_2\rangle =&\langle a_1^2\rangle \cos\theta \sin\theta + \langle a_2^2\rangle(-\sin\theta\cos\theta)\\ =&\left(\langle a_1^2\rangle - \langle a_2^2\rangle\right)\cos\theta\sin\theta\end{aligned}$$

Requires that θ is zero for any $\theta \rightarrow$ the parentheses must be zero:

$$\langle a_1^2\rangle = \langle a_2^2\rangle$$

That is, the equipartition theorem.

(c) Prove that only the exponential distribution satisfies rotation invariance.

Given:

$$\rho(a_1, a_2) = f\left(a_1^2\right) f\left(a_2^2\right)$$

After rotation:

$$f\left(a_1^2\right) f\left(a_2^2\right) = f\left(b_1^2\right) f\left(b_2^2\right)$$

Let:

$$f(x) = e^{-g(x)}.$$

Then:

$$g\left(a_1^2\right) + g\left(a_2^2\right) = g\left(b_1^2\right) + g\left(b_2^2\right).$$

Also considering $a_1^2 + a_2^2 = b_1^2 + b_2^2$ Therefore, there is only one form:

$$g(x) \propto x.$$

To require that it holds for any θ, it can only be true according to $a_1^2 + a_2^2$. Standard generalization proves that the unique function f satisfying the equation is an exponential function:

$$f\left(a^2\right) = e^{-\beta a^2/2}$$

This is the Gaussian form of the Maxwell distribution.

(d) Using the definition $\frac{1}{2}\langle a^2\rangle = kT$ to find β

Calculation:

$$\langle a^2\rangle = \frac{\int a^2 e^{-\beta a^2/2} da}{\int e^{-\beta a^2/2} da} = \frac{2}{\beta}$$

Due to the definition:

$$\begin{aligned}\frac{1}{2}\langle a^2\rangle &= kT\\ \frac{1}{2}\frac{2}{\beta} &= kT\\ \beta &= \frac{1}{kT}\end{aligned}$$

This recovers the Maxwell-Boltzmann distribution.

Problem 3 — Atoms escaping a furnaced

Particles facing the furnace have the following velocity distribution:

$$dn = nAvf(v)dv$$

This follows a three-dimensional Maxwell distribution:

$$f(\mathbf{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m\left(v_x^2+v_y^2+v_z^2\right)}{2kT}\right)$$

For a particle to escape, $v_z > 0$, the number of atoms passing through the area A per second (with velocities from $\mathbf{v}$ to $\mathbf{v}+d\mathbf{v}$) is given, and only particles with $v_z > 0$ can escape:

$$dn = Anf(\mathbf{v})v_z d^3v, \quad v_z > 0$$

In spherical coordinates:

$$dn = An\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} v^3 (\cos\theta \sin\theta) dv d\theta d\phi$$

Integrating over the angle yields the velocity distribution of the escaping particles

$$dn(v) = An\frac{m}{2\pi kT} v^3 \exp\left(-\frac{mv^2}{2kT}\right) dv$$

This is the standard outflow distribution, multiplied by an additional term v compared to the furnace flow (because faster particles are more likely to hit the orifice).

(b) RMS velocity of outflowing particles

$$v\_\mathrm{rms}^{(e)} = \sqrt{\frac{\int v^5 e^{-mv^2/2kT} dv}{\int v^3 e^{-mv^2/2kT} dv}}$$

Using Gaussian integral:

$$v\_\mathrm{rms}^{(e)} = \sqrt{\frac{4kT}{m}}$$

RMS inside the furnace:

$$v\_\mathrm{rms}^{(i)} = \sqrt{\frac{3kT}{m}}$$

Therefore:

$$v * \mathrm{rms}^{(e)} = \sqrt{\frac{4}{3}} v * \mathrm{rms}^{(i)}$$

The outflowing particles tend to be faster, because faster particles are more likely to collide with the hole.

Problem 4 — Blackbody Radiation

$$I(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1}$$
$$z = \frac{\hbar\omega}{kT} \quad \Rightarrow \quad \omega = z\frac{kT}{\hbar}$$

(a) Total Radiation Intensity $\propto$ T^4 (Stefan-Boltzmann Law)

Integral:

$$I = \int_0^\infty I(\omega) d\omega$$

Variable Transformation:

$$d\omega = \frac{kT}{\hbar} dz$$

Substitute:

$$I = \frac{\hbar}{\pi^2 c^3} \int_0^\infty \frac{(zkT/\hbar)^3}{e^z - 1} \frac{kT}{\hbar} dz$$

Extract T:

$$I = \frac{(kT)^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{z^3}{e^z - 1} dz$$

The integral is a constant, so:

$$I \propto T^4$$

(b) Maximum frequency $\omega_m \propto T$ (Wien's displacement law) Find the extrema:

$$\frac{dI}{d\omega} = 0 \qquad \text{where } I(z) \propto \frac{z^3}{e^z - 1}$$

The maximum value satisfies:

$$3\left(1 - e^{-z}\right) - z = 0$$

Solving for:

$$z_m \approx 2.821$$

Back to frequency:

$$\omega_m = z_m \frac{kT}{\hbar} \propto T$$