TA homework5-2

Kepler's Problem (Complete Derivation)

Let the central force be $\mathbf{F} = -\frac{\gamma}{r^2}\hat{e}_r$. The mass of the point mass is m. The potential energy is taken as:

$$U(r) = -\frac{\gamma}{r}$$

Kinetic energy $T = \frac{1}{2}mv^2$. Total Energy

$$E = T + U = \frac{1}{2}mv^2 - \frac{\gamma}{r}$$

Angular Momentum Scalar

$$l = mr^2\dot{\theta}$$

For ease of writing, we define specific energy and specific angular momentum:

$$\varepsilon = \frac{E}{m}, \quad h = \frac{l}{m}$$

Note: The lemma consistent with common notation is $\mu = \frac{\gamma}{m}$, therefore the specific potential is $-\frac{\mu}{r}$.

1) Express the standard relationship (specific energy form) of the semi-major axis a (for an ellipse, $E < 0$) in terms of E and l:

$$\varepsilon = \frac{E}{m} = -\frac{\mu}{2a} \quad \Longrightarrow \quad a = -\frac{\mu}{2\varepsilon}.$$

Using back $\mu = \gamma/m, \varepsilon = E/m$:

$$a = -\frac{\gamma}{2E}$$

(Note: Because $E < 0$, the expression gives a positive a.) The standard formula for the eccentricity e (for inverse square potential):

$$e^2 = 1 + \frac{2\varepsilon h^2}{\mu^2}$$

Substituting $\varepsilon = E/m, h = l/m, \mu = \gamma/m$:

$$e^2 = 1 + \frac{2(E/m)\left(l^2/m^2\right)}{(\gamma^2/m^2)} = 1 + \frac{2El^2}{m\gamma^2}$$

Therefore

$$e = \sqrt{1 + \frac{2El^2}{m\gamma^2}}$$

Semi-minor axis b using a, e :

$$b = a\sqrt{1 - e^2}$$

Substituting into the above equation, it can be written as a combination of a and l, E: first write $1 - e^2 = -\frac{2El^2}{m\gamma^2}$ Therefore,

$$b^2 = a^2\left(1 - e^2\right) = a^2\left(-\frac{2El^2}{m\gamma^2}\right)$$

More concisely, the following c expression yields a more elegant form (see below). The length of the half-chord c (in the problem: "the half-chord passing through the focus and perpendicular to the major axis") is geometrically provable (or can be proven by substituting into the equation of the ellipse to find the intersection point):

$$c = \frac{b^2}{a} = a\left(1 - e^2\right)$$

Substituting $a=-\frac{\gamma}{2E}$ and $1-e^2=-\frac{2El^2}{m\gamma^2}$ into the equation:

$$c=a\left(1-e^2\right)=\left(-\frac{\gamma}{2E}\right)\left(-\frac{2El^2}{m\gamma^2}\right)=\frac{l^2}{m\gamma}$$

Therefore, it is very concise:

$$c=\frac{l^2}{m\gamma}$$

Additionally, from $b^2=ac$ (obtained from $b^2=a^2\left(1-e^2\right)$ and $c=a\left(1-e^2\right)$), can be written as:

$$\boxed{b=\sqrt{ac}=\sqrt{a\cdot\frac{l^2}{m\gamma}}}$$

Checking the "specialness" of a and c

From the above expression:

$$\begin{aligned}
a&=-\frac{\gamma}{2E} \quad \text{depends only on } E(\text{ and constant } \gamma),\\
c&=\frac{l^2}{m\gamma} \quad \text{depends only on } l(\text{ and constant } m,\gamma).
\end{aligned}$$

Therefore a is determined only by energy, and c is determined only by angular momentum—this completes the proof.

(3) Which orbit maximizes l when E is fixed? Which orbit minimizes E when l is fixed?

From $e^2=1+\frac{2El^2}{m\gamma^2}$.

If $E<0$ is fixed, the maximum value of l is required such that the orbit remains an ellipse (i.e., $e^2\geq 0$ and $e<1$). The physical boundary is when the ellipse becomes a circle

$(e=0)$, reaching its extreme value. Let $e=0$, then

$$0=1+\frac{2El_{\max}^2}{m\gamma^2}\Rightarrow l_{\max}^2=-\frac{m\gamma^2}{2E}$$

Therefore,

$$l_{\max}=\gamma\sqrt{\frac{m}{-2E}},$$

\$0.1 The corresponding orbit is a circular orbit. Intuitively: for a given energy, to maximize angular momentum, the orbit needs to be the most symmetrical—a circle.

If l is fixed, to minimize E (i.e., make it the most negative), again refer to the above equation: e^2 cannot be negative, and the minimum E corresponds to $e=0$ (otherwise E would be closer to 0). Therefore,

$$0=1+\frac{2E_{\min}l^2}{m\gamma^2}\Rightarrow E_{\min}=-\frac{m\gamma^2}{2l^2}$$

Thus, when l is fixed, the circular orbit has the minimum energy (most stable/lowest energy state).

Kepler's Third Law (General Elliptical Orbits) For a two-body problem with its center of mass at the center, the specific momentum $\mu=\frac{\gamma}{m}$. The standard form of Kepler's Third Law (period T) (using the gravitational parameter μ) is:

$$T^2=\frac{4\pi^2a^3}{\mu}.$$

Substituting $\mu=\gamma/m$ back:

$$T^2=\frac{4\pi^2ma^3}{\gamma}$$

If we use the common celestial mechanics notation $\tilde{\mu} = G(M_{\text{center}}) = \gamma/m$ (for example, $\gamma = GMm$), then the known form is:

$$T^2 = \frac{4\pi^2 a^3}{G(M_{\text{center}})}.$$

Problem 2 — Cosmic velocities

(1) First Cosmic Velocity v_1 (Minimum speed required for the Earth to orbit the Earth in a circular path)

The orbital velocity for the Earth's surface in a circular path is given by the balance between centripetal force and gravitational force:

$$\frac{mv_1^2}{R_\oplus} = \frac{GM_\oplus m}{R_\oplus^2} \quad \Rightarrow \quad v_1 = \sqrt{\frac{GM_\oplus}{R_\oplus}}$$

(That is, the circular velocity in the near-Earth orbit.)

(2) Second cosmic velocity v_2 (Minimum speed required to escape from the Earth's surface to infinity)

Escape velocity (energy that makes the specific energy 0):

$$\frac{1}{2}v_2^2 - \frac{GM_\oplus}{R_\oplus} = 0 \quad \Rightarrow \quad v_2 = \sqrt{\frac{2GM_\oplus}{R_\oplus}} = \sqrt{2}v_1$$

(3) Third cosmic velocity v_3 (capable of escaping the solar system from Earth's surface)

Physical idea (simplified model):To escape the Sun's gravitational well, from a heliocentric perspective, the spacecraft's velocity in the heliocentric system after launch must at least reach the Sun's local escape velocity at Earth's orbit: $v_{\text{esc,Sun}} = \sqrt{2GM_\odot/r}$. However, Earth itself has an orbital velocity around the Sun: $v_E = \sqrt{GM_\odot/r}$. Therefore, if the spacecraft is launched from Earth in the same direction as Earth's revolution (prograde motion), its velocity in the heliocentric system is...

$$v_{\text{sc, Sun}} = v_{\text{launch}} + v_E$$

(vector addition, prograde motion is added); if launched in the opposite direction, the values are subtracted. To minimize the launch velocity (velocity relative to Earth) required for launch from Earth's surface, prograde motion is often chosen to utilize Earth's kinetic energy.

However, it is still necessary to overcome Earth's gravitational well (i.e., escape from Earth's surface first). A commonly used approximate synthetic energy method (combining the Earth well and heliocentric escape conditions) is given (prograde motion is most favorable):

$$v_3 \approx \sqrt{v_2^2 + (v_{\text{esc,Sun}} - v_E)^2}$$

Substituting $v_2 = \sqrt{2GM_\oplus/R_\oplus}$, $v_{\text{esc, Sun}} = \sqrt{2}v_E$, then

$$v_3 \approx \sqrt{v_2^2 + (\sqrt{2}-1)^2 v_E^2}$$

Substituting the values (Earth orbital velocity $v_E \approx 29.78\ \text{km/s}$, Earth escape velocity $v_2 \approx 11.2\ \text{km/s}$) to get

$$v_3 \approx 16.7\ \text{km/s}$$

This is the value of the third cosmic velocity

(4) Using a more "strict" energy method in the heliocentric frame

Write the total (specific) energy at the launch point in the heliocentric frame: specific energy at the moment of launch

$$\varepsilon_{\text{init}} = \frac{1}{2}(v_{\text{launch}} + v_E)^2 - \frac{GM_\odot}{r} - \frac{GM_\oplus}{R_\oplus},$$

If it is required that the solar system can eventually be escaped (specific energy at a distance ≥ 0), and the potential energy of the Earth can be reduced to zero (i.e., the Earth well has been overcome), then the required value can be solved. v_{launch} . Treating Earth's potential energy as the quantity that needs to be overcome first, after simplification, an approximate expression equivalent to the above synthesis formula can be obtained. The tedious algebra will not be elaborated here—the synthesis energy formula given is already a commonly used and physically clear result.

(5) Liu's misunderstanding in the novel (regarding "If the speed of light is reduced to v_c, the solar system will become a black hole")

This passage clearly confuses two concepts: the speed of light cannot "shrink" in the classical Newtonian/relativity sense to lead to the formation of a black hole—a black hole is formed by concentrating gravity (mass/energy) into a sufficiently small radius (event horizon), the radius of which depends on the mass M and the value of the speed of light c in a vacuum (Schwarzschild radius $r_s = 2GM/c^2$). However, using "reducing the speed of light to a certain value" as a means is not equivalent to concentrating mass within a scale—and relativity assumes that the speed of light is constant and the reference frame remains unchanged. In short, equating "the speed of light slowing down" with "the system turning into a black hole" in the novel is a misinterpretation of relativity and black hole theory.

Problem 3 — DART Impact (Numerical Estimation and Comparison)

(1) Assuming a circular orbit: Calculate the orbital radius R and orbital velocity v. By Kepler's third law for two bodies (with $a = R$):

$$T^2 = \frac{4\pi^2 R^3}{G\,(M_1 + m_2)} \quad \Rightarrow \quad R = \left(\frac{G\,(M_1 + m_2)T^2}{4\pi^2}\right)^{1/3}$$

Algebraic calculation (substituting the values) yields $R \approx \mathbf{1.184 \times 10^3\ m}$ (approximately 1184 m, i.e., orbital velocity).

The orbital radius is approximately one kilometer—this illustrates that satellite orbits in a binary system are very close! Orbital velocity (circular orbit):

$$v = \sqrt{\frac{G\,(M_1 + m_2)}{R}} \approx \mathbf{0.1733\ m/s(\approx 0.624\ km/h).}$$

In other words, Dimorphos orbits Didymos at a very low speed (a fitting analogy to "turtle speed").

2) Spacecraft colliding with a completely inelastic (stick) collision: How to design to maximize the shortening of the orbital period? Compare estimates with NASA's values

Optimal design: To maximize the shortening of the orbital period, the orbital energy after the collision should be minimized—this requires projecting impact/retrograde motion relative to the orbital direction, reducing the orbital tangential velocity). Furthermore, it is desirable to maximize the impact velocity and mass, and to have the impact occur close to the orbital tangent to maximize the change in tangential velocity. Additionally, generating a large amount of projectiles that are ejected in the opposite direction to the orbital trajectory would give the target object additional recoil (momentum boost), resulting in a larger change in angular momentum than simple inelastic adhesion (this is the so-called "momentum boosting factor" β, see below).

Simple rigid body perfectly inelastic estimation (directly using momentum conservation): Assume the impact direction is exactly opposite to the orbital velocity (the most favorable direction). The system velocity after the collision (combined mass $m_2 + m_s$) is:

$$v' = \frac{m_2 v - m_s v_s}{m_2 + m_s}$$

Because $m_2 \gg m_s$, the velocity reduction can be approximated by a small amount:

$$\Delta v \approx \frac{m_s}{m_2} v_s$$

Substituting the values (previously calculated as $v \approx \mathbf{0.1733\ m/s},\ v_s \approx \mathbf{6258.33\ m/s},\ m_2 = \mathbf{4.8 \times 10^9\ kg},\ m_s = 600$ kg The numerical results are as follows:

- The decrease in orbital velocity caused by the collision (contribution from pure momentum adhesion) is approximately:

$$\Delta v \approx 7.82 \times 10^{-4}\ \mathrm{m/s}$$

The period change is derived from the velocity change (assuming an initial circular orbit, the orbit after the collision is approximately still near-circular, using energy relations):
Initial specific energy $\varepsilon_i = \frac{1}{2}v^2 - \frac{\mu}{R} = -\frac{\mu}{2R}$. After the collision, the velocity becomes $v_{\text{new}} = v - \Delta v$, and the new specific energy $\varepsilon_f = \frac{1}{2}v_{\text{new}}^2 - \frac{\mu}{R}$. The new period T_f can be calculated from $a = -\frac{\mu}{2\varepsilon}$ and $T = 2\pi a^{3/2}/\sqrt{\mu}$. Substituting the values, we get:

- Initial period $T_i = 42900\ \mathrm{s}$ (consistent with the given value).
- The period change estimated above is approximately

$$\Delta T = T_f - T_i \approx -573\ \mathrm{s} \approx -9.55\ \mathrm{minutes.}$$

If we assume complete inelastic adhesion (no ejecta) and all momentum is contributed solely by the ship's mass, the period will shorten by approximately 9.6 minutes.