TA homework4

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Appendix

Conic Sections

All conic sections can be defined geometrically as follows: For a fixed point (the **focus**) and a fixed line (the **directrix**), any point P on the curve satisfies that the ratio of its distance to the focus and its distance to the directrix d is a constant :

$$\frac{r}{d} = e$$

The constant e is called the **eccentricity**.

Cutting Plane Orientation	Resulting Curve	Name	Characteristic	Relation between p and a
Plane perpendicular to cone axis	Circle	$e = 0$	Focus coincides with center	$p = a$
Plane slightly tilted but not through base	Ellipse	$0 < e < 1$	Closed orbit	$p = a\left(1 - e^2\right)$
Plane parallel to cone's generatrix	Parabola	$e = 1$	Critical case (escape orbit)	$p = a\left(1 - e^2\right)$, but $a \to \infty, e = 1$
Plane more inclined, cutting both nappes	Hyperbola	$e > 1$	Open curve (fly-by trajectory)	$p = a\left(e^2 - 1\right)$

$$r(\theta) = \frac{p}{1 + e\cos\theta}$$

In a central-force (gravitational) system, we take the focus (the Sun) as the origin.

- r : distance to the focus
- θ : true anomaly, the angle from periapsis
- p : semi-latus rectum, setting the size
- e : eccentricity, setting the shape
  For any conic orbit governed by an inverse-square law, the following **dynamical relations** hold:

$$\begin{cases} p = \frac{h^2}{\mu}, \\ e^2 = 1 + \frac{2\varepsilon h^2}{\mu^2}, \end{cases}$$

where

- $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$ is the **specific energy,**
- $h = r^2\dot{\theta}$ is the **specific angular momentum,**
- $\mu = GM$ is the **gravitational parameter.**

Eccentricity – Energy Relation

Defining the specific energy as $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$, one can derive directly from the orbital solution that

$$e^2 = 1 + \frac{2\varepsilon h^2}{\mu^2}$$

Proof
From $p = h^2/\mu$ and $r = \frac{p}{1+e\cos\theta}$, the velocity components are

$$\begin{cases} v_r = \dot{r} = \frac{he}{p}\sin\theta \\ v_\theta = r\dot{\theta} = \frac{h}{r} = \frac{h(1+e\cos\theta)}{p} \end{cases} \implies v^2 = \frac{\mu^2}{h^2}\left(1 + e^2 + 2e\cos\theta\right).$$

then

$$\varepsilon=\frac{v^2}{2}-\frac{\mu}{r}=\frac{\mu^2}{2h^2}\left(1+e^2+2e\cos\theta\right)-\frac{\mu(1+e\cos\theta)}{p}$$

using $p=h^2/\mu$ then the second term becomes $\frac{\mu^2}{h^2}(1+e\cos\theta)$, and

$$\varepsilon=\frac{\mu^2}{2h^2}\left(e^2-1\right). \implies \boxed{e^2=1+\frac{2\varepsilon h^2}{\mu^2}}$$

From this, the corresponding relationship between the above-mentioned energy and the orbital category can be immediately obtained.

Hodograph (Velocity Space) Analytical Expression and Proof that "Velocity endpoints are circles"

We start from the orbital equation to find the velocity component and construct the circular equation in the velocity space： from $r(\theta)=\frac{p}{1+e\cos\theta}$ ， and $p=h^2/\mu$ 。 first, calculate the radial and angular components of the velocity：

we can get

$$v_r=\dot{r}=\frac{dr}{d\theta}\dot{\theta}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{h}{r^2}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot hu^2=-h\frac{du}{d\theta}.$$

substitute $u=(1/r)=\frac{1+e\cos\theta}{p}$ then we can get

$$v_r=\dot{r}=\frac{he}{p}\sin\theta$$

Tangential velocity (component around the Angle)

$$v_\theta=r\dot{\theta}=\frac{h}{r}=\frac{h(1+e\cos\theta)}{p}.$$

By combining and using $h/p=\mu/h$, the component expression of the velocity vector is obtained:

$$\mathbf{v}(\theta)=v_r\hat{\mathbf{r}}+v_\theta\hat{\boldsymbol{\theta}}=\dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\theta}=\frac{\mu}{h}(e\sin\theta\hat{\mathbf{r}}+(1+e\cos\theta)\hat{\boldsymbol{\theta}}).$$

Now, in the velocity space, if $v_\theta$is regarded as a horizontal coordinate and $v_r$as a rigid coordinate, then

$$\left(v_\theta-\frac{\mu}{h}\right)^2+v_r^2=\left(\frac{\mu}{h}e\cos\theta\right)^2+\left(\frac{\mu}{h}e\sin\theta\right)^2=\left(\frac{\mu e}{h}\right)^2,$$

This is a circle with point $(\mu/h,0)$as its center and radius $\mu e/h$. Therefore, the trajectory of the velocity endpoints is a circle - this is the analytical proof of hodograph. Based on the relative positions of the center of the circle to the origin (whether the circle contains the origin or passes through it), the geometric judgment of $e>1, e=1, and e<1$is obtained, which is consistent with the conclusions of questions 1 and 2.

Problem 4

Let the central mass be M, and the gravitational constant mu=G M. Take the mass of the particle and the polar coordinates (r,θ). The inverse square force is

$$\mathbf{F}=-\frac{m\mu}{r^2}\hat{\mathbf{r}}$$

The polar coordinate equation of motion (radial and angular) of a particle is

$$m\left(\ddot{r}-r\dot{\theta}^2\right)=-\frac{m\mu}{r^2},\quad m(r\ddot{\theta}+2\dot{r}\dot{\theta})=0$$

The angular equation gives the conservation of angular momentum: define the angular momentum $h$per unit mass (specific angular momentum

$$h\equiv r^2\dot{\theta}=\text{ Const.}\implies\dot{\theta}=\frac{h}{r^2}$$

代换 $u(\theta)=1/r$ 并异出微分方程

Let $u(\theta)=\frac{1}{r}$. We need to express $\ddot{r}-r\dot{\theta}^2$with respect to θ. First, find the derivative of $r$with respect to θ:

$$r=\frac{1}{u},\quad \frac{dr}{d\theta}=-\frac{1}{u^2}\frac{du}{d\theta}.$$

Convert the time derivative to the angular derivative using the chain rule:

$$\dot{r}=\frac{dr}{d\theta}\dot{\theta}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{h}{r^2}\xrightarrow{r^2=1/u^2}\dot{r}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot hu^2=-h\frac{du}{d\theta}.$$

Then find the second derivative:

$$\ddot{r}=-h\frac{d}{dt}\left(\frac{du}{d\theta}\right)=-h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\dot{\theta}=-h\frac{d^2u}{d\theta^2}\cdot\dot{\theta}.$$

Using $\dot{\theta}=h/r^2=hu^2$, we get

$$\ddot{r}=-h\frac{d^2u}{d\theta^2}\cdot hu^2=-h^2u^2\frac{d^2u}{d\theta^2}.$$

Now calculate the second term $r\dot{\theta}^2$:

$$r\dot{\theta}^2=\frac{1}{u}\left(h^2u^4\right)=h^2u^3$$

then

$$\ddot{r}-r\dot{\theta}^2=-h^2u^2\frac{d^2u}{d\theta^2}-h^2u^3=-h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)$$

Divide both sides of the radial equation by m and substitute them into the above equation (and $1/r^2=u^2$):

$$m\left(\ddot{r}-r\dot{\theta}^2\right)=-\frac{m\mu}{r^2}\xrightarrow{\ddot{r}-r\dot{\theta}^2=-h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)}\boxed{\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}}$$

Solve the equation

The general solution to the homogeneous equation $u''+u=0$ is $A\cos(\theta)+B\sin(\theta)$. We take a constant solution as a particular solution $u_p=\frac{\mu}{h^2}$. Therefore, the overall solution is:

$$u(\theta)=\frac{\mu}{h^2}+A\cos\theta+B\sin\theta$$

Let the constants A and B be expressed in polar form as eccentricity: Let

$$A=e\frac{\mu}{h^2}\cos\theta_0,\quad B=e\frac{\mu}{h^2}\sin\theta_0$$

Then

$$u(\theta)=\frac{\mu}{h^2}(1+e\cos(\theta-\theta_0))$$

Let the semi-bore $\boxed{p=\frac{h^2}{\mu}}$, and substitute back $r=1/u$:

$$r(\theta)=\frac{p}{1+e\cos\theta_0\,(\theta-\theta_0)}$$

This is the polar equation of a conic section. The parameter e is the eccentricity, which determines the type of curve:

- $0\leq e<1$: Ellipse (bound, $\varepsilon<0$).
- $e=1$: Parabola (critical, $\varepsilon=0$).
- $e>1$: Hyperbola (unbound, $\varepsilon>0$).

Common simplification: Taking the closest point to the apex as $\theta=\theta_0$, we can set θ_0 to 0, obtaining $r=p/(1+e\cos\theta)$.

Problem 1

Kepler

Consider a point mass m with position vector $\mathbf{r}$ and velocity $\mathbf{v}=\dot{\mathbf{r}}$. The characteristic of a central force is that the force $\mathbf{F}$ is always directed along $\mathbf{r}$ (i.e., $\mathbf{F}=f(r)\hat{\mathbf{r}}$). Consequently, the torque is:

$$\boldsymbol{\tau}=\mathbf{r}\times\mathbf{F}=\mathbf{r}\times(f(r)\hat{\mathbf{r}})=\mathbf{0}$$

Due to the conservation of angular momentum:

$$\mathbf{L}=\mathbf{r}\times m\mathbf{v}=\text{ Const.}$$

The relationship between the magnitude of the angular momentum and the areal velocity: In polar coordinates, the instantaneous areal velocity (the area swept per unit time) is:

$$\frac{dA}{dt}=\frac{1}{2}r^2\dot{\theta}\overset{L=mr^2\dot{\theta}}{=\!=\!=\!=}\frac{dA}{dt}=\frac{L}{2m}=\text{ Constant.}$$

This is the Law of Equal Areas, which is Kepler's Second Law.****

velocity space circle

For a inverse-square ($\propto 1/r^2$) central force, it can be proven that the endpoint of the velocity vector traces a circle in velocity space (called a hodograph).

Let $\mathbf{r}(t)$ be the position and $\mathbf{v}(t)$ be the velocity of the particle at time t. Over a small time interval Δt, the force is the central force $\mathbf{F}=f(r)\hat{\mathbf{r}}$. The impulse (change in velocity per unit mass) is:

$$\Delta\mathbf{v}\approx\frac{\mathbf{F}}{m}\Delta t=\frac{f(r)}{m}\hat{\mathbf{r}}\Delta t$$

Using the conservation of angular momentum, the angular momentum per unit mass, $h=r^2\dot{\theta}$, is a constant. The corresponding change in angle over the same Δt is approximately $\Delta\theta\approx\boldsymbol{\dot{\theta}\Delta t}=\frac{h}{r^2}\boldsymbol{\Delta t}\Longrightarrow\boldsymbol{\Delta t}=\frac{r^2}{h}\boldsymbol{\Delta\theta}$. Substituting $\boldsymbol{\Delta t}$ into $\boldsymbol{\Delta\mathbf{v}}$:

$$\Delta\mathbf{v}\approx\frac{f(r)}{m}\hat{\mathbf{r}}\cdot\frac{r^2}{h}\Delta\theta=\frac{r^2f(r)}{mh}\hat{\mathbf{r}}\Delta\theta$$

For a general central force, there is still a dependence on r. However, when the force is an inverse-square force, $f(r)=-\frac{\mu m}{r^2}$ (attraction) or $f(r)=+\frac{\mu m}{r^2}$ (repulsion). For simplicity, we first discuss the attractive force case (Newton's Universal Gravitation):

$$\Delta\mathbf{v}\approx\frac{r^2f(r)}{mh}\hat{\mathbf{r}}\Delta\theta\approx\frac{r^2\cdot\left(\pm\mu m/r^2\right)}{mh}\hat{\mathbf{r}}\Delta\theta=\pm\frac{\mu}{h}\hat{\mathbf{r}}\Delta\theta\Longrightarrow\boxed{\frac{d\mathbf{v}}{d\theta}=\pm\frac{\mu}{h}\hat{\mathbf{r}}(\theta)}$$

$$\Delta\mathbf{v}\approx\frac{r^2f(r)}{mh}\hat{\mathbf{r}}\Delta\theta\approx\frac{r^2\cdot\left(-\mu m/r^2\right)}{mh}\hat{\mathbf{r}}\Delta\theta=-\frac{\mu}{h}\hat{\mathbf{r}}\Delta\theta\Longrightarrow\frac{d\mathbf{v}}{d\theta}=-\frac{\mu}{h}\hat{\mathbf{r}}(\theta)$$

For the inverse-square force, the length of the velocity increment is independent of r, depending only on the fixed constant μ/h and the angle increment $\Delta\theta$. Integrating both sides with respect to θ (where $\hat{\mathbf{r}}(\theta)=\cos\theta\cdot\hat{\mathbf{x}}+\sin\theta\cdot\hat{\mathbf{y}}$ and $\int\hat{\mathbf{r}}(\theta)d\theta=\hat{\boldsymbol{\theta}}(\theta)+\mathbf{C}'$):

$$\mathbf{v}(\theta)=\underbrace{\mathbf{v}\left(\theta_0\right)+\frac{\mu}{h}\hat{\boldsymbol{\theta}}\left(\theta_0\right)}_{\text{Constant vector A}}-\underbrace{\frac{\mu}{h}}\cdot\hat{\boldsymbol{\theta}}(\theta)$$

Taking the magnitude of the vector difference:

$$\left|\mathbf{v}(\theta)+\frac{\mu}{h}\hat{\boldsymbol{\theta}}(\theta)\right|=|\mathbf{A}|=\text{ Const.}$$

The magnitude of the constant vector $\mathbf{A}$, $|\mathbf{A}|=\mu e/h$, is the radius of the shifted circle. At the periapsis (closest point): The radial velocity ; the tangential velocity $v_\theta=\frac{h}{r_{\min}}=\frac{\mu}{h}(1+e)$; the direction $\hat{\boldsymbol{\theta}}$ is perpendicular to $\hat{\mathbf{r}}$. The magnitude of the velocity vector is:

$$v_p=\frac{\mu}{h}(1+e).$$

On the other hand, the distance from the velocity endpoint on the hodograph to the center of the circle is always μ/h, and the distance from the origin (zero velocity point) to the center of the circle is $|\mathbf{A}|$. So the length of the velocity endpoint from the origin at periapsis is:

$$v_p = |\mathbf{A}| + \frac{\mu}{h}$$

Substitute the above $v_p = \frac{\mu}{h}(1+e)$:

$$\frac{\mu}{h}(1+e) = |\mathbf{A}| + \frac{\mu}{h} \implies |\mathbf{A}| = \frac{\mu e}{h}$$

Thus obtaining

$$\left(v_\theta - \frac{\mu}{h}\right)^2 + v_r^2 = \left(\frac{\mu e}{h}\right)^2$$

- In the velocity space, if the circle corresponding to the velocity trajectory encloges the origin inside, the corresponding eccentricity $e > 1$(hyperbolic).
- If the circle passes exactly through the origin, then (parabolic);
- If the origin is outside the circle, then $e < 1$(ellipse).

Problem 2

from the result the hodograph,

$$\left(v_\theta - \frac{\mu}{h}\right)^2 + v_r^2 = \left(\frac{\mu e}{h}\right)^2$$

To test whether the hodograph circle passes through the origin $(v_\theta, v_r) = (0, 0)$ (i.e. $lim_{r\to\infty}\mathbf{v} = 0$ the critical case) plug the origin into :

$$\left(0 - \frac{\mu}{h}\right)^2 + 0^2 = \left(\frac{\mu e}{h}\right)^2 \iff \left(\frac{\mu}{h}\right)^2 = \left(\frac{\mu e}{h}\right)^2 \iff e^2 = 1.$$

which mean it a hyperbola corresonding with $e = 1$ thus the circle:

$$\textbf{Parabola} \iff \underline{e=1} \iff \underline{\varepsilon = 0} \iff \textbf{hodograph passes through the origin.}$$

Problem 3

Equal area ratio

Same as question 1: As long as the force is radial (whether attractive or repulsive), the torque $\mathbf{r} \times \mathbf{F} = 0$, so angular momentum is conserved, and consequently, the area velocity is constant. Therefore, Kepler's second law (equal area ratio) still holds. The repulsive/attractive sign does not affect this conclusion.

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{constant}.$$

b is the **collision parameter** (the perpendicular distance between the initial direction of motion of the incident particle and the target nucleus), v_0 is the initial velocity at infinity.

$$L = mr^2\dot{\theta} = \mu b v_0$$

Orbit Type

In Rutherford scattering, the incident particle is incident at high speed, $\varepsilon = \frac{1}{2}\frac{M}{m+M}v^2 + \frac{1}{m}\frac{kq_1q_2}{r^2} > 0$, from which the eccentricity relationship can be obtained:

$$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}} > 1$$

The orbit is one branch of a hyperbola (the particle comes from infinity, passes close, and then returns to infinity).

Similarly, the position of the hodograph circle can be determined in velocity space: the origin is located inside the circle, at infinity ($\lim_{r\to\infty} v_r \neq 0, v_\theta = 0$). According to the hodograph equations, this is a trajectory containing the origin, which is a hyperbolic orbit.

$$\left(v_\theta + \frac{\mu}{h}\right)^2 + v_r^2 = \left(\frac{\mu e}{h}\right)^2.$$

Central Force Form (Uniqueness of the Inverse Square Law)

A planet revolves around the sun in a plane, its orbit being elliptical, with the sun at the focus. Let the orbital equation be:

$$r = \frac{p}{1 + e\cos\theta}, \quad e < 1.$$

Assume conservation of angular momentum: $h = r^2\dot{\theta}$.

1. Derive the functional form that the central force $F(r)$ must satisfy to cause the particle to move along this orbit;

First, find $\dot{r}$ and $\ddot{r}$ (the derivatives with respect to time), but using θ as the independent variable, make $u' \equiv du/d\theta$, $u'' \equiv d^2u/d\theta^2$.

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{d(1/u)}{d\theta}\dot{\theta} = -\frac{1}{u^2}u' \cdot \left(hu^2\right) = -hu'$$

$$\Downarrow$$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{d}{dt}\left(-hu'\right) = -h\frac{du'}{dt} = -h\frac{du'}{d\theta}\dot{\theta} = -hu''\left(hu^2\right) = -h^2u^2u'',$$

Then calculate $r\dot{\theta}^2$:

$$r\dot{\theta}^2 = \frac{1}{u}\left(h^2u^4\right) = h^2u^3$$

then we can get the radial acceleration:

$$\ddot{r} - r\dot{\theta}^2 = -h^2u^2u'' - h^2u^3 = -h^2u^2\left(u'' + u\right).$$

From newton's law

$$-mh^2u^2\left(u'' + u\right) = F(r) \tag{*}$$

This step is the key general formula: for any central force $F(r)$, the right-hand side holds, and all θ depends on $u(\theta)$. If the orbit is given as $r(\theta) = \frac{p}{1+e\cos\theta}$, then

$$u(\theta) = \frac{1}{r} = \frac{1 + e\cos\theta}{p} = \frac{1}{p} + \frac{e}{p}\cos\theta.$$

then we can get

$$u'' + u = -\frac{e}{p}\cos\theta + \frac{1}{p} + \frac{e}{p}\cos\theta = \frac{1}{p}$$

Substitude to $(*)$ we can get

$$F(r) = -mh^2u^2 \cdot \frac{1}{p} = -\frac{mh^2}{p}u^2 = -\frac{mh^2}{p} \cdot \frac{1}{r^2} = \boxed{\frac{m\mu}{r^2}}$$