1 Problem 1:

Second-Order Taylor Expansion

The new length of the spring is:

$$L_1 = \sqrt{(a-x)^2 + (0-y)^2}$$

Or consider a function:

$$f(x,y) = \sqrt{(a-x)^2 + y^2}$$

The second-order Taylor expansion of a binary function $f(x,y)$ around (x_0, y_0) is:

$$\begin{aligned} f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) &+ \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \Delta y \\ &+ \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} (\Delta x)^2 + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} \Delta x \Delta y + \frac{1}{2}\left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} (\Delta y)^2 \end{aligned}$$

Here, $f(x,y) = L_1(x,y)$, $(x_0,y_0) = (0,0)$, and $\Delta x = x$, $\Delta y = y$ (displacements from equilibrium).

Substitute $(x,y) = (0,0)$ into $L_1(x,y)$:

$$f(0,0) = L_1(0,0) = \sqrt{(a-0)^2 + 0^2} = a$$

First, rewrite $L_1(x,y)$ as $[(a-x)^2 + y^2]^{1/2}$ for differentiation.

Using the chain rule $\frac{\partial}{\partial x}[g(x,y)]^{1/2} = \frac{1}{2}[g(x,y)]^{-1/2} \cdot \frac{\partial g}{\partial x}$, where $g(x,y) = (a-x)^2 + y^2$:

$$\frac{\partial L_1}{\partial x} = \frac{1}{2}\left[(a-x)^2 + y^2\right]^{-1/2} \cdot (-2)(a-x) = -\frac{a-x}{\sqrt{(a-x)^2+y^2}}$$

$$\left.\frac{\partial L_1}{\partial x}\right|_{(0,0)} = -\frac{a-0}{\sqrt{a^2+0^2}} = -1$$

Similarly,

$$\frac{\partial L_1}{\partial y} = \frac{1}{2}\left[(a-x)^2 + y^2\right]^{-1/2} \cdot 2y = \frac{y}{\sqrt{(a-x)^2+y^2}}$$

$$\left.\frac{\partial L_1}{\partial y}\right|_{(0,0)} = \frac{0}{\sqrt{a^2+0^2}} = 0$$

$$\frac{\partial^2 L_1}{\partial x^2} = \frac{y^2}{\left[(a-x)^2+y^2\right]^{3/2}}$$

Substitute $(x,y) = (0,0)$:

$$\left.\frac{\partial^2 L_1}{\partial x^2}\right|_{(0,0)} = \frac{0^2}{a^3} = 0$$

$$\frac{\partial^2 L_1}{\partial y^2} = \frac{(a-x)^2}{\left[(a-x)^2+y^2\right]^{3/2}}$$

$$\left.\frac{\partial^2 L_1}{\partial y^2}\right|_{(0,0)} = \frac{a^2}{a^3} = \frac{1}{a}$$

Differentiate $\frac{\partial L_1}{\partial x} = -\frac{a-x}{\sqrt{(a-x)^2+y^2}}$ with respect to y:

$$\frac{\partial^2 L_1}{\partial x \partial y} = -(a-x) \cdot \left(-\frac{1}{2}\right) \left[(a-x)^2 + y^2\right]^{-3/2} \cdot 2y = \frac{y(a-x)}{\left[(a-x)^2 + y^2\right]^{3/2}}$$

$$\left. \frac{\partial^2 L_1}{\partial x \partial y} \right|_{(0,0)} = \frac{0 \cdot a}{a^3} = 0$$

We get:

$$\begin{aligned} L_1(x,y) &\approx a + (-1) \cdot x + 0 \cdot y + \frac{1}{2} \cdot 0 \cdot x^2 + 0 \cdot xy + \frac{1}{2} \cdot \frac{1}{a} \cdot y^2 \\ &= a - x + \frac{y^2}{2a} \end{aligned}$$

Or, more naive,

$$a\sqrt{1 - \frac{2x}{a} + \frac{x^2+y^2}{a^2}} \approx a(1 + \frac{1}{2}(-\frac{2x}{a} + \frac{x^2+y^2}{a^2}) - \frac{1}{8}(-\frac{2x}{a} + \frac{x^2+y^2}{a^2})^2) \approx a - x + \frac{y^2}{2a}$$

Calculate Potential Energy U

Based on this, we get $\Delta l_1 = L_1 - l_0 = (a - x + \frac{y^2}{2a}) - l_0 = (a - l_0) - x + \frac{y^2}{2a}$

$$\begin{aligned} U_1 &= \frac{1}{2}k \left[(a - l_0) - x + \frac{y^2}{2a}\right]^2 \\ &\approx \frac{1}{2}k(a-l_0)^2 - k(a-l_0)x + \frac{k(a-l_0)y^2}{2a} + \frac{1}{2}kx^2 \end{aligned} \tag{1}$$

$$\begin{aligned} U_{\text{all}} &= (U_1 + U_2) + (U_3 + U_4) \\ &= 2k(a-l_0)^2 + \left[kx^2 + \frac{k(a-l_0)y^2}{a}\right] + \left[ky^2 + \frac{k(a-l_0)x^2}{a}\right] \\ &= 2k(a-l_0)^2 + kx^2\left(1 + \frac{a-l_0}{a}\right) + ky^2\left(1 + \frac{a-l_0}{a}\right) \end{aligned} \tag{2}$$

We don't care constant here, so$U_{\text{all}} = k \cdot \frac{2a-l_0}{a}(x^2+y^2) = \frac{1}{2} \cdot \underbrace{\frac{2k(2a-l_0)}{a}}_{k'} r^2 = \frac{1}{2}k'r^2$

Force

$\vec{F} = -\nabla U$,so we get $F_x = -\frac{\partial U_{\text{all}}}{\partial x} = -k'x$,similarly $F_y = -k'y$.

2 Problem 2

We have $m\ddot{z}(t) = \rho_w g S(A\sin(\omega t) - z(t))$, consider $k = \rho_w g S, \omega_0^2 = \frac{k}{m}$, we get the expression for classical forced vibration:

$$\ddot{z}(t) + \omega_0^2 z(t) = \frac{\rho_w g S A \sin(\omega t)}{m}, w_o^2 = \frac{g}{L}$$

. The stable solution is $z(t) = Z\sin(\omega t - \phi), Z = \frac{\frac{gA}{L}}{\omega_0^2 - \omega^2}$. Now we can write description:

1. $\omega < \omega_0, \phi = 0$,two wave in same phase.
2. $\omega > \omega_0, \phi = \pi$,two wave in opposite phase.
3. $\omega = \omega_0$, resonance.

Problem 3

1(a) $F(t) = F_0 for t \geq 0$

We need solve:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \tag{1}$$

The homogeneous equation ($F(t) = 0$) has solutions:

$$x_h(t) = e^{-\beta t}\left(C_1 \cos\omega t + C_2 \sin\omega t\right)$$

$\beta = \gamma/2$ and $\omega = \sqrt{\omega_0^2 - \beta^2}$. $Tosimplifythequestion, hereweonlydiscuss \gamma/2 < \omega$ For constant $F(t) = F_0$, assume a constant particular solution $x_p(t) = A$, we have

$$0 + 0 + \omega_0^2 A = \frac{F_0}{m} \implies A = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

Initial conditions:

$$x(0) = 0, \quad \dot{x}(0) = 0$$

$x(0) = 0$:

$$0 = e^0\left(C_1 \cos 0 + C_2 \sin 0\right) + \frac{F_0}{k} \implies C_1 = -\frac{F_0}{k}$$

$\dot{x}(0) = 0$:

$$\dot{x}(t) = -\beta e^{-\beta t}(C_1 \cos\omega t + C_2 \sin\omega t) + e^{-\beta t}(-\omega C_1 \sin\omega t + \omega C_2 \cos\omega t)$$

$$0 = -\beta C_1 + \omega C_2 \implies C_2 = \frac{\beta}{\omega}C_1 = -\frac{\beta F_0}{k\omega}$$

Final Solution

$$x(t) = \frac{F_0}{k}\left[1 - e^{-\beta t}\left(\cos\omega t + \frac{\beta}{\omega}\sin\omega t\right)\right] \tag{2}$$

1(b) $F(t) = F_0 \cos\omega_0 t$ for $t \geq 0$

Homogeneous Solution Same as 1(a): $x_h(t) = e^{-\beta t}(C_1 \cos\omega t + C_2 \sin\omega t)$.

Particular Solution Assume $x_p(t) = B\cos\omega_0 t + D\sin\omega_0 t$. Substitute into (1):
Compute derivatives:

$$\dot{x}_p = -\omega_0 B\sin\omega_0 t + \omega_0 D\cos\omega_0 t$$

$$\ddot{x}_p = -\omega_0^2 B\cos\omega_0 t - \omega_0^2 D\sin\omega_0 t$$

Substitute into (1) with $F(t)/m = F_0\cos\omega_0 t/m$:

$$\begin{aligned}&(-\omega_0^2 B\cos\omega_0 t - \omega_0^2 D\sin\omega_0 t)\\ &+\gamma(-\omega_0 B\sin\omega_0 t + \omega_0 D\cos\omega_0 t)\\ &+\omega_0^2(B\cos\omega_0 t + D\sin\omega_0 t) = \frac{F_0}{m}\cos\omega_0 t\end{aligned}$$

Simplify (cancel ω_0^2 terms):

$$\gamma\omega_0 D\cos\omega_0 t - \gamma\omega_0 B\sin\omega_0 t = \frac{F_0}{m}\cos\omega_0 t$$

Equate coefficients:
- $\cos\omega_0 t$: $\gamma\omega_0 D = F_0/m \implies D = \frac{F_0}{m\gamma\omega_0}$
- $\sin\omega_0 t$: $-\gamma\omega_0 B = 0 \implies B = 0$

Thus, $x_p(t) = \frac{F_0}{m\gamma\omega_0}\sin\omega_0 t$.

Complete Solution and Initial Conditions Complete solution: $x(t) = e^{-\beta t}(C_1\cos\omega t + C_2\sin\omega t) + \frac{F_0}{m\gamma\omega_0}\sin\omega_0 t$.
Apply $x(0) = 0$:

$$0 = C_1 + 0 \implies C_1 = 0$$

Differentiate $x(t)$:

$$\dot{x}(t) = e^{-\beta t}\left[-\beta C_2\sin\omega t + \omega C_2\cos\omega t\right] + \frac{F_0\omega_0}{m\gamma\omega_0}\cos\omega_0 t$$

Apply $\dot{x}(0) = 0$:

$$0 = \omega C_2 + \frac{F_0}{m\gamma} \implies C_2 = -\frac{F_0}{m\gamma\omega}$$

Final Solution

$$x(t) = \frac{F_0}{m\gamma}\left(\frac{\sin\omega_0 t}{\omega_0} - \frac{e^{-\beta t}\sin\omega t}{\omega}\right) \tag{3}$$

Part 2: Frequency for Maximum Amplitude

For long times, transient terms ($e^{-\beta t}$) vanish. For $F = F_0\cos\omega t$, assume steady-state solution:

$$x_p(t) = A(\omega)\cos(\omega t - \phi)$$

Substitute into (1),we have

$$\left[-A\omega^2\cos(\omega t - \phi)\right] + \gamma\left[-A\omega\sin(\omega t - \phi)\right] + \omega_0^2\left[A\cos(\omega t - \phi)\right] = \frac{F_0}{m}\cos\omega t$$

Group terms by $\cos(\omega t - \phi)$ and $\sin(\omega t - \phi)$:

$$A\left(\omega_0^2 - \omega^2\right)\cos(\omega t - \phi) - A\gamma\omega\sin(\omega t - \phi) = \frac{F_0}{m}\cos\omega t \tag{4}$$

Use the cosine addition formula:

$$\cos\omega t = \cos(\omega t - \phi)\cos\phi - \sin(\omega t - \phi)\sin\phi$$

Substitute into Eq. (4):

$$\begin{aligned} &A\left(\omega_0^2 - \omega^2\right)\cos(\omega t - \phi) - A\gamma\omega\sin(\omega t - \phi) \\ &= \frac{F_0}{m}\left[\cos(\omega t - \phi)\cos\phi - \sin(\omega t - \phi)\sin\phi\right] \end{aligned}$$

For equality to hold for all t, coefficients of $\cos(\omega t - \phi)$ and $\sin(\omega t - \phi)$ must match on both sides:

1. Coefficient of $\cos(\omega t - \phi)$:

$$A\left(\omega_0^2 - \omega^2\right) = \frac{F_0}{m}\cos\phi \tag{5}$$

2. Coefficient of $\sin(\omega t - \phi)$:

$$-A\gamma\omega = -\frac{F_0}{m}\sin\phi \implies A\gamma\omega = \frac{F_0}{m}\sin\phi \tag{6}$$

So we have:

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \tag{7}$$

To maximize $A(\omega)$, minimize the denominator $D(\omega) = (\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2$.
Take derivative $dD/d\omega = 0$:

$$2(\omega_0^2 - \omega^2)(-2\omega) + 2\gamma^2\omega = 0$$

$$-4\omega(\omega_0^2 - \omega^2) + 2\gamma^2\omega = 0$$

For $\omega \neq 0$, divide by 2ω:

$$-2(\omega_0^2 - \omega^2) + \gamma^2 = 0 \implies \omega^2 = \omega_0^2 - \frac{\gamma^2}{2}$$

Thus, the resonance frequency is:

$$\omega^* = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} \tag{8}$$

3 Problem 4

1. Instantaneous Power and Average Power

Instantaneous Power $P(t)$

$P(t) = F(t) \cdot \dot{x}(t)$.

$$\dot{x}(t) = -A\omega \sin(\omega t - \phi)$$

$F(t) = F_0 \cos \omega t$, we have:

$$P(t) = F_0 \cos \omega t \cdot [-A\omega \sin(\omega t - \phi)]$$

$$= -F_0 A\omega \cos \omega t \sin(\omega t - \phi)$$

Using $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$:

$$P(t) = -F_0 A\omega \cos \omega t \left[\sin \omega t \cos \phi - \cos \omega t \sin \phi\right]$$

$$= -F_0 A\omega \cos \phi \cdot \cos \omega t \sin \omega t + F_0 A\omega \sin \phi \cdot \cos^2 \omega t$$

Average Power $\langle P \rangle$

The average over one period $T = 2\pi/\omega$ is:

$$\langle P \rangle = \frac{1}{T} \int_0^T P(t) dt$$

Using trigonometric averages over a full cycle:

$$\langle \cos \omega t \sin \omega t \rangle = \langle \frac{1}{2} \sin 2\omega t \rangle = 0$$

$$\langle \cos^2 \omega t \rangle = \frac{1}{2}$$

Only the second term survives:

$$\langle P \rangle = F_0 A\omega \sin \phi \cdot \frac{1}{2}$$

For a damped oscillator, the phase shift satisfies $\sin \phi = \frac{2\beta\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$ and amplitude $A = \frac{F_0/(2m\beta\omega)}{\sin \phi}$ (derived from steady-state analysis). Substituting these:

$$\langle P \rangle = F_0 \cdot \frac{F_0}{2m\beta\omega \sin \phi} \cdot \omega \sin \phi \cdot \frac{1}{2} \cdot 2m\beta\omega^2 \cdot \frac{F_0}{2m\beta\omega \sin \phi} \cdot \sin \phi$$

Simplifying (using $F_0 = 2m\beta\omega A \sin \phi$ from amplitude-phase relations):

$$\boxed{\langle P \rangle = m\beta\omega^2 A^2}$$

2. Resistive Force and Power

The resistive force is $f_{\text{res}} = -2m\beta\dot{x}(t)$, so:

$$P_{\text{res}}(t) = |f_{\text{res}} \cdot \dot{x}(t)| = 2m\beta\dot{x}(t)^2$$

The average dissipation rate is:

$$\langle P_{\text{res}} \rangle = 2m\beta\langle \dot{x}(t)^2 \rangle$$

Substituting $\dot{x}(t) = -A\omega \sin(\omega t - \phi)$ and using $\langle \sin^2(\omega t - \phi) \rangle = \frac{1}{2}$:

$$\langle P_{\text{res}} \rangle = 2m\beta \cdot A^2\omega^2 \cdot \frac{1}{2} = m\beta\omega^2 A^2$$

It matches $\langle P_{\text{res}} \rangle = \langle P \rangle$.

3. Resonance of Average Power

From question 1, $\langle P \rangle = m\beta\omega^2 A^2$. Substituting the amplitude $A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+(2\beta\omega)^2}}$:

$$\langle P \rangle = m\beta\omega^2 \cdot \frac{F_0^2/m^2}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2} = \frac{\beta F_0^2\omega^2}{m\left[(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2\right]}$$

To maximize $\langle P \rangle$, consider the denominator $D(\omega) = (\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2$. Take the derivative $\frac{dP}{d\omega} = 0$, or we write $f(\omega) = \cdot\frac{\omega^2}{D(\omega)}$, we have:

$$f'(\omega) = \frac{2\omega D(\omega) - \omega^2 D'(\omega)}{[D(\omega)]^2}$$

$$D'(\omega) = -4\omega(\omega_0^2 - \omega^2) + 8\beta^2\omega = 4\omega\left[-(\omega_0^2 - \omega^2) + 2\beta^2\right]$$

For $f'(\omega) = 0$,

$$\begin{aligned}
0 &= 2\omega D(\omega) - \omega^2 D'(\omega) \\
&= 2\omega((\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2) - \omega^2(4\omega(-(\omega_0^2 - \omega^2) + 2\beta^2)) \\
&= (\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2 - 2\omega^2(-\omega_0^2 + \omega^2 + 2\beta^2) \\
&= \omega_0^4 - \omega^4
\end{aligned}$$

For $\omega \neq 0$, the maximum occurs at $\omega^* = \omega_0$: