Exercises and explanations of matrix multiplication, calculus, and ordinary differential equations (ODE)

matrix multiplication

The definition of a matrix

The $m \cdot n$ numbers are arranged into a table with m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{1}$$

This is called an $m \cdot n$ matrix.
When $m = n$, matrix A is referred to as an n^{th} order matrix or an n^{th} order square matrix.

The scalar multiplication of a matrix

The multiplication of a scalar k and a matrix A is defined as:

$$kA = k\left[a_{ij}\right] = \left[ka_{ij}\right] \tag{2}$$

Question 1:

Let $A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & -4 & 5 \end{pmatrix}$, find $3A$.

Solution1:

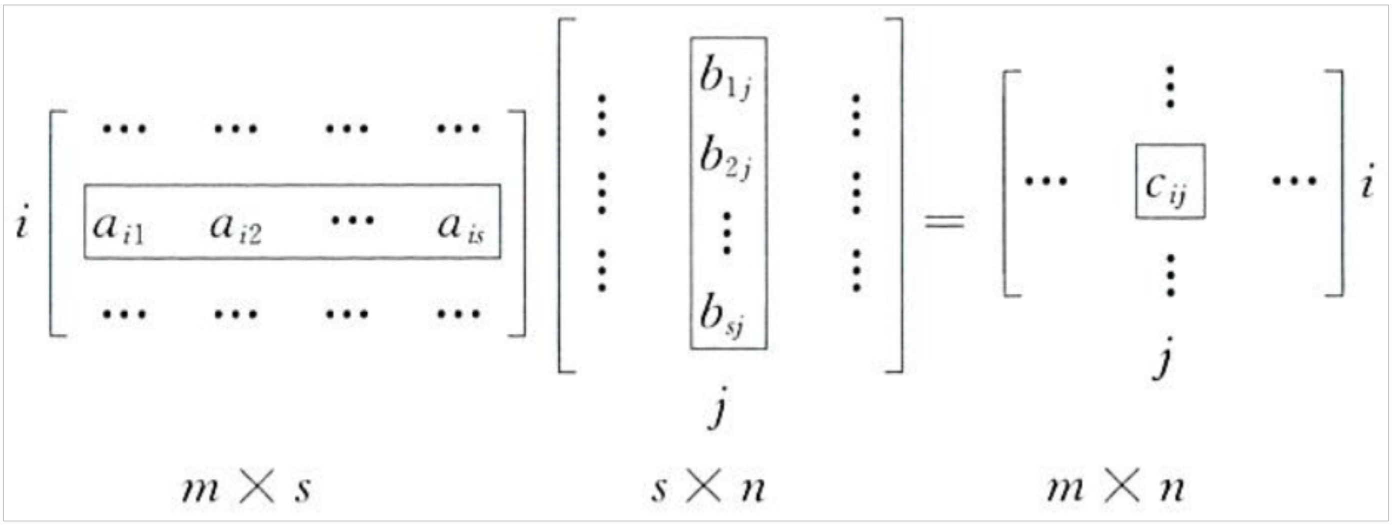
$$3A = \begin{pmatrix} 3 \cdot 1 & 3 \cdot 3 & 3 \cdot (-2) \\ 3 \cdot 0 & 3 \cdot (-4) & 3 \cdot 5 \end{pmatrix} = \begin{pmatrix} 3 & 9 & -6 \\ 0 & -12 & 15 \end{pmatrix} \tag{3}$$

The multiplication of matrices

Let A be an $m \cdot s$ matrix and B be an $s \cdot n$ matrix. Then the product AB is an $m \cdot n$ matrix.
Denote it as $C = AB = [C_{ij}]_{m \cdot n}$, where the element C_{ij} of the i^{th} row and the j^{th} column of C is the sum of the products of the s elements in the i^{th} row of A and the corresponding s elements in the j^{th} column of B.

$$C_{ij} = \sum_{k=1}^{s} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{is}b_{sj} \tag{4}$$

The diagram of matrix multiplication：



Specifically, let A be an $n \cdot n$ square matrix. Then, the notation $A \cdot A \cdot \cdots \cdot A = A^k$ is referred to as the k^{th} power of A.

Question 2:

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & -4 \\ -5 & 2 \end{pmatrix} \tag{5}$$

Find:

1. AB
2. BA
3. A^2

Solution 2:

For AB:

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 10 & -4 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 10 + 2 \cdot (-5) & 1 \cdot (-4) + 2 \cdot 2 \\ 3 \cdot 10 + 6 \cdot (-5) & 3 \cdot (-4) + 6 \cdot 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

For BA:

$$BA = \begin{pmatrix} 10 & -4 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 10 \cdot 1 + (-4) \cdot 3 & 10 \cdot 2 + (-4) \cdot 6 \\ (-5) \cdot 1 + 2 \cdot 3 & (-5) \cdot 2 + 2 \cdot 6 \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \tag{7}$$

For A^2:

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 6 \\ 3 \cdot 1 + 6 \cdot 3 & 3 \cdot 2 + 6 \cdot 6 \end{pmatrix} = \begin{pmatrix} 7 & 14 \\ 21 & 42 \end{pmatrix} = 7A \tag{8}$$

Question 3:

Let the matrices be defined as follows:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

Find:

1. $AB = ?, BA = ?$
2. $AC = ?, CA = ?$
3. $CB = ?, BC = ?$
4. $A^2 = ?, B^2 = ?, C^2 = ?$

Solution 3:

For AB:

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} (i \cdot 1) + (0 \cdot 0) & (0 \cdot -i) + (1 \cdot 0) \\ (i \cdot 0) + (1 \cdot -i) & (0 \cdot 0) + (i \cdot 1) \end{pmatrix} = \begin{pmatrix} i & 0 \\ -i & 0 \end{pmatrix} \tag{10}$$

For BA:

$$BA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (0 \cdot 0) + (-i \cdot 1) & (0 \cdot 1) + (-i \cdot 0) \\ (i \cdot 0) + (0 \cdot 1) & (i \cdot 1) + (0 \cdot 0) \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{11}$$

For AC:

$$AC = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} (0 \cdot 1) + (1 \cdot 0) & (0 \cdot 0) + (1 \cdot -1) \\ (1 \cdot 1) + (0 \cdot 0) & (1 \cdot 0) + (0 \cdot -1) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

For CA:

$$CA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (1 \cdot 0) + (0 \cdot 1) & (1 \cdot 1) + (0 \cdot 0) \\ (0 \cdot 0) + (-1 \cdot 1) & (0 \cdot 1) + (-1 \cdot 0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{13}$$

For CB:

$$CB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} (1 \cdot 0) + (0 \cdot i) & (1 \cdot -i) + (0 \cdot 0) \\ (0 \cdot 0) + (-1 \cdot i) & (0 \cdot -i) + (-1 \cdot 0) \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{14}$$

For BC:

$$BC = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} (0 \cdot 1) + (-i \cdot 0) & (0 \cdot 0) + (-i \cdot -1) \\ (i \cdot 1) + (0 \cdot 0) & (i \cdot 0) + (0 \cdot -1) \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{15}$$

For A^2:

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (0 \cdot 0) + (1 \cdot 1) & (0 \cdot 1) + (1 \cdot 0) \\ (1 \cdot 0) + (0 \cdot 1) & (1 \cdot 1) + (0 \cdot 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{16}$$

For B^2:

$$B^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} (0 \cdot 0) + (-i \cdot i) & (0 \cdot -i) + (-i \cdot 0) \\ (i \cdot 0) + (0 \cdot i) & (i \cdot -i) + (0 \cdot 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{17}$$

For C^2:

$$C^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} (1 \cdot 1) + (0 \cdot 0) & (1 \cdot 0) + (0 \cdot -1) \\ (0 \cdot 1) + (-1 \cdot 0) & (0 \cdot 0) + (-1 \cdot -1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{18}$$

Thus, $A^2 = B^2 = C^2 = I$.

Question 4:

Let the matrices be:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{19}$$

Find:

1. $AB = ?$
2. $BA = ?$
3. $AC = ?$
4. $CA = ?$
5. $CB = ?$
6. $BC = ?$
7. $A^2 = ?$
8. $B^2 = ?$
9. $C^2 = ?$

Solution 4:

- For AB:

$$AB = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix} \tag{20}$$

- For BA:

$$BA = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & i \end{pmatrix} \tag{21}$$

- For AC:

$$AC = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

- For CA:

$$CA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \tag{23}$$

- For CB:

$$CB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix} \tag{24}$$

- For BC:

$$BC = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & i \\ 0 & 0 & 0 \end{pmatrix} \tag{25}$$

- For A^2:

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{26}$$

- For B^2:

$$B^2 = B \cdot B = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{27}$$

- For C^2:

$$C^2 = C \cdot C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{28}$$

Question 5:

Let the matrices be:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{29}$$

Find:

1. $AB - BA$
2. $(AB)^2$
3. A^2B^2

Solution 5:

- For $AB - BA$:

First, compute AB:

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 3 & 1 \cdot 2 + 0 \cdot 4 \\ 0 \cdot 1 + (-1) \cdot 3 & 0 \cdot 2 + (-1) \cdot 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \tag{30}$$

Next, compute BA:

$$BA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 & 1 \cdot 0 + 2 \cdot (-1) \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \tag{31}$$

Now, compute $AB - BA$:

$$AB - BA = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 1 - 1 & 2 - (-2) \\ -3 - 3 & -4 - (-4) \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -6 & 0 \end{pmatrix} \tag{32}$$

- For $(AB)^2$

First, we already know that $AB = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$.

Now, compute $(AB)^2 = AB \cdot AB$:

$$(AB)^2 = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 1-6 & 2-8 \\ -3+12 & -6+16 \end{pmatrix} = \begin{pmatrix} -5 & -6 \\ 9 & 10 \end{pmatrix} \tag{33}$$

- For A^2B^2

First, compute A^2 and B^2.

For A^2:

$$A^2 = A \times A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{34}$$

For B^2:

$$B^2 = B \times B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 1 + 4 \cdot 3 & 3 \cdot 2 + 4 \cdot 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} \tag{35}$$

Now, compute A^2B^2:

$$A^2B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} \tag{36}$$

Thus, $A^2B^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$.

calculus

The General Form of a Taylor Series

The general form of a Taylor series for a function $f(x)$ expanded around a point $x = a$ is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \tag{37}$$

Alternatively, the general form can be written as an infinite sum:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{38}$$

Here:

- $f^{(n)}(a)$ is the n-th derivative of $f(x)$ evaluated at $x = a$,
- $n!$ denotes n factorial,
- $(x - a)$ is the distance from the expansion point a.

Question 1:

Given the function $ f(x) = \sin(x) $:

(a) Find the Taylor expansion of $ \sin(x) $ at $x = 0$ and write out the first three terms.

(b) Using the first three terms of the Taylor expansion, approximate $ \sin(0.1) $, and compare the result with the actual value (give the actual value to four decimal places).

Solution 1:

(a)

The general form of a Taylor series is:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \tag{39}$$

For $\sin(x)$, we calculate its derivatives:

- $f(x) = \sin(x)$, so $f(0) = 0$
- $f'(x) = \cos(x)$, so $f'(0) = 1$
- $f''(x) = -\sin(x)$, so $f''(0) = 0$
- $f'''(x) = -\cos(x)$, so $f'''(0) = -1$

Substituting these into the Taylor series gives the first three terms for $\sin(x)$ at $x = 0$:

$$\sin(x) \approx x - \frac{x^3}{3!} \tag{40}$$

which simplifies to:

$$\sin(x) \approx x - \frac{x^3}{6} \tag{41}$$

(b)

Substitute $x = 0.1$:

$$\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{6} = 0.1 - \frac{0.001}{6} = 0.1 - 0.0001667 = 0.0998333 \tag{42}$$

The actual value is $\sin(0.1) \approx 0.09983$, so the approximation is very close.

Question 2:

Given the function $ f(x) = \ln(1+x) $:

(a) Find the Taylor expansion of $ \ln(1+x) $ at $x = 0$ and write out the first three terms.

(b) Using the first three terms of the Taylor expansion, approximate $ \ln(1+0.2) $, and compare the result with the actual value (give the actual value to four decimal places).

Solution 2:

(a)

The general form of a Taylor series is:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \tag{43}$$

For $f(x) = \ln(1+x)$, we calculate its derivatives:

- $f(0) = \ln(1+0) = 0$
- $f'(x) = \frac{1}{1+x}$, so $f'(0) = 1$
- $f''(x) = -\frac{1}{(1+x)^2}$, so $f''(0) = -1$
- $f'''(x) = \frac{2}{(1+x)^3}$, so $f'''(0) = 2$

Substituting these into the Taylor series gives:

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \tag{44}$$

(b)

Substitute $x = 0.2$:

$$\ln(1+0.2) \approx 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} \tag{45}$$

$$= 0.2 - \frac{0.04}{2} + \frac{0.008}{3} = 0.2 - 0.02 + 0.00267 = 0.18267 \tag{46}$$

The actual value is $\ln(1.2) \approx 0.18232$, so the approximation is quite close.

Question 3:

Given the function $ f(x) = e^x $:

(a) Find the Taylor expansion of $ e^x $ at $x = 0$ and write out the first three terms.

(b) Using the first three terms of the Taylor expansion, approximate $e^{0.2}$, and compare the result with the actual value (give the actual value to four decimal places).

Solution 3:

(a)

For $f(x) = e^x$, all the derivatives of $f(x)$ are equal to $e^x $, and at $x = 0$, $f^{(n)}(0) = 1$ for all n. The Taylor series expansion is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{47}$$

(b)

Substitute $x = 0.2$:

$$e^{0.2} \approx 1 + 0.2 + \frac{(0.2)^2}{2!} + \frac{(0.2)^3}{3!} \tag{48}$$

$$= 1 + 0.2 + \frac{0.04}{2} + \frac{0.008}{6} = 1 + 0.2 + 0.02 + 0.00133 = 1.22133 \tag{49}$$

The actual value is $e^{0.2} \approx 1.22140$, so the approximation is very accurate.

Here are the common 8 Taylor series expansions for frequently encountered functions:

1. **Exponential Function $ e^x $**:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{50}$$

2. **Sine Function $ \sin(x) $**:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{51}$$

3. **Cosine Function** $\cos(x)$:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{52}$$

4. **Tangent Function** $\tan(x)$ (for $|x| < \frac{\pi}{2}$):

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots \tag{53}$$

5. **Natural Logarithm $ \ln(1+x) $** (for $|x| < 1$):

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \tag{54}$$

6. **Binomial Series** $\frac{1}{1-x}$ (for $|x| < 1$):

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \tag{55}$$

7. **Binomial Series** $\frac{1}{1+x}$ (for $|x| < 1$):

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \tag{56}$$

8. **Binomial Series** $(1+x)^n$ (for $|x| < 1$):

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots \tag{57}$$

Exercise lesson 1

Practices and solutions of vector multiplication, calculus calculation and ODE exercises.

Calculus calculation 微积分计算

Indefinite Integral

$$\int f(x)dx = F(x) + C$$

Definite Integral

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Basic Integral Formulas 基本积分公式

(1) $\int 0 \mathrm{d}x = C$;

(2) $\int 1 \mathrm{d}x = \int \mathrm{d}x = x + C$;

(3) $\int x^\alpha \mathrm{d}x = \frac{1}{\alpha+1}x^{\alpha+1} + C(\alpha \neq -1)$,

(4) $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$,

(5) $\int a^x \mathrm{d}x = \frac{a^x}{\ln a} + C(a > 0, a \neq 1)$,

(6) $\int \mathrm{e}^x \mathrm{d}x = \mathrm{e}^x + C$,

(7) $\int \sin x \mathrm{d}x = -\cos x + C$,

(8) $\int \cos x \mathrm{d}x = \sin x + C$

(9) $\int \tan x \mathrm{d}x = -\ln|\cos x| + C$,

(10) $\int \cot x \mathrm{d}x = \ln|\sin x| + C$,

(11) $\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C$,

(12) $\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C$,

(13) $\int \sec^2 x \mathrm{d}x = \tan x + C$,

(14) $\int \csc^2 x \mathrm{d}x = -\cot x + C$,

(15) $\int \frac{1}{a^2+x^2}\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$,

(16) $\int \frac{1}{a^2-x^2}\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$,

(17) $\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin\frac{x}{a} + C$,

(18) $\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$.

Example 1

$$\int e^x(1-\frac{e^{-x}}{\sqrt{x}})dx = \int e^x dx - \int x^{-\frac{1}{2}}dx = e^x - 2x^{\frac{1}{2}} + C$$

$$\int cos^2\frac{x}{2}dx = \int \frac{1+cosx}{2}dx = \frac{x+sinx}{2} + C$$

$$\int (\frac{3}{1+x^2} - \frac{2}{\sqrt{1-x^2}})dx = 3\int \frac{dx}{1+x^2} - 2\int \frac{dx}{\sqrt{1-x^2}} = 3arctanx - 2arcsinx + C$$

The Method of Substitution for Integration 换元积分法

Let $u = g(x)$, so $du = g'(x)dx$.

Indefinite integrals

$$\int f(x)dx = \int f(g(u)) \cdot g'(u)du$$

Definite integrals

$$\int_a^b f(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 2

Simple substitution for Integration

Let $u = x^2$, then $du = 2xdx$

$$\int 2xe^{x^2}dx = \int e^u du = e^u + C = e^{x^2} + C$$

The substitution method for trigonometric functions

Let $u = 3x$, then $du = 3dx$, so $dx = \frac{du}{3}$.

$$\int \sin 3x dx = \int \sin u \cdot \frac{du}{3} = \frac{1}{3}\int \sin u du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos 3x + C$$

Integration by Parts 分部积分法

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Example 3

$$\int xe^x dx = xe^x - \int 1 \cdot e^x dx = xe^x - e^x + C$$

$$\int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C$$

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$$

Improper integrals 反常积分

1. Improper Integrals on an Infinite Interval

$$\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

2. Improper Integrals with a Discontinuity or Singularity

$$\int_0^1 \frac{1}{x^2} dx = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{x^2} dx$$

An important improper integral

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = 2 \int_0^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Ordinary Differential Equations (ODEs)

Ordinary Differential Equations (ODEs) are equations that describe the relationship between an unknown function and its derivatives. A differential equation where the unknown function is a function of one variable is an ordinary differential equation.

The general form

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

First-Order ODEs 一阶微分方程

1. **Differential equations with separable variables** 变量可分离的微分方程

   Example 1

   $$\frac{dy}{dx} = xy$$

   Solution

   $$\frac{1}{y}dy = xdx$$
   $$\int \frac{1}{y}dy = \int xdx$$
   $$\ln|y| = \frac{x^2}{2} + C$$

   The general solution to the equation

   $$y = Ce^{x^2/2}$$

2. **Homogeneous differential equation** 齐次微分方程

   $$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

   Usually, the equation is simplified by introducing the variable substitution $v = \frac{y}{x}$, after which it is transformed into a method that can separate variables to solve.

   Example 2

   $$\frac{dy}{dx} = \frac{x+y}{x}$$

   Solution

   $$\frac{dy}{dx} = 1 + \frac{y}{x}$$

Set $v = \frac{y}{x}$, then $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$$v + x\frac{dv}{dx} = 1 + v$$
$$x\frac{dv}{dx} = 1$$

Separate variables and integrate

$$\int dv = \int \frac{1}{x}dx$$
$$v = \ln|x| + C$$
$$\frac{y}{x} = \ln|x| + C$$
$$y = x(\ln|x| + C)$$

3. **First-order linear differential equation** 一阶线性微分方程

$$y' + p(x)y = q(x)$$

general solution

$$y = e^{-\int p(x)dx}[\int q(x)e^{\int p(x)dx}dx + C]$$

4. **Bernoulli's Equation** 伯努利方程

$$y' + p(x)y = q(x)y^n$$

convert to

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

Second-order ODEs 二阶微分方程

A second-order linear differential equation has the general form:

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

If $g(x) = 0$, the equation is called a **homogeneous second-order differential equation.**

If $g(x) \neq 0$, it is called a **non-homogeneous** equation.

1. **Constant Coefficient Second-Order Homogeneous Differential Equations 常系数二阶线性齐次微分方程**

$$ay'' + by' + cy = 0$$

Characteristic equation

$$ar^2 + br + c = 0$$

- Two unequal real roots $r_1 \neq r_2$, the general solution is:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

- Two equal real roots $r_1 = r_2 = r$, the general solution is:

$$y(x) = (C_1 + C_2 x) e^{rx}$$

- Two conjugate complex roots $r_{1,2} = \alpha \pm \beta i$, the general solution is:

$$y(x) = e^{\alpha x} \left(C_1 \cos \beta x + C_2 \sin \beta x \right)$$

Example 3 Solving a Second-Order Linear Homogeneous Equation with Constant Coefficients

$$y'' - 3y' + 2y = 0$$

Characteristic equation

$$r^2 - 3r + 2 = 0$$

Solve the characteristic equation

$$(r-1)(r-2) = 0$$

The roots are $r_1 = 1$ and $r_2 = 2$.

Write the general solution

Since the characteristic equation has two distinct real roots, the general solution is

$$y(x) = C_1 e^x + C_2 e^{2x}$$

Example 4 Solving a Second-Order Linear Homogeneous Equation with Complex Roots

$$y'' + 4y = 0$$

Characteristic equation

$$r^2 + 4 = 0$$

Solve the characteristic equation

$$r = \pm 2i$$

Write the general solution

Since the roots are complex, the general solution is

$$y(x) = C_1 \cos 2x + C_2 \sin 2x$$

2. **Non-Homogeneous Second-Order Differential Equations 非齐次二阶微分方程**

   For non-homogeneous equations of the form $ay'' + by' + cy = g(x)$, the general solution is the sum of the solution to the homogeneous equation $y_h(x)$ and a particular solution $y_p(x)$

$$y(x) = y_h(x) + y_p(x)$$

   where $y_h(x)$ is the general solution to the corresponding homogeneous equation, and $y_p(x)$ is a particular solution to the non-homogeneous equation.

   Methods such as **variation of parameters** and **undetermined coefficients** are commonly used to find the particular solution.

   Example 5 Solving a Non-Homogeneous Second-Order Linear Equation

$$y'' - 3y' + 2y = e^x$$

   Solve the homogeneous equation

First, solve the homogeneous equation $y'' - 3y' + 2y = 0$. The characteristic equation is $r^2 - 3r + 2 = 0$, with roots $r_1 = 1$ and $r_2 = 2$. So the general solution to the homogeneous equation is

$$y_h(x) = C_1 e^x + C_2 e^{2x}$$

Find a particular solution

After substituting and solving, we find that the particular solution is $y_p(x) = xe^x$.

Write the final general solution

$$y(x) = y_h(x) + y_p(x) = C_1 e^x + C_2 e^{2x} + xe^x$$

Example 6 Non-Homogeneous Term with Trigonometric Functions

$$y'' + 4y = \sin 2x$$

Find the general solution to the homogeneous equation

The corresponding homogeneous equation is

$$y'' + 4y = 0$$

The characteristic equation is

$$r^2 + 4 = 0$$

Solving, we get $r = \pm 2i$. So the general solution to the homogeneous equation is:

$$y_h(x) = C_1 \cos 2x + C_2 \sin 2x$$

Find the particular solution

The non-homogeneous term is $g(x) = \sin 2x$. We assume the particular solution has the form:

$$y_p(x) = A \cos 2x + B \sin 2x$$

Substituting this into the equation and solving gives $A = 0$, $B = \frac{1}{4}$. Therefore, the particular solution is:

$$y_p(x) = \frac{1}{4} \sin 2x$$

Write the general solution

The total solution is

$$y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4} \sin 2x$$

定理（某些特殊自由项的二阶常系数线性非齐次微分方程的解法）

类型 1　方程 $y'' + py' + qy = P_m(x)e^{ax}$ 的解法（其中 $P_m(x)$ 为 x 的 m 次已知多项式）：

第一步，写出对应的齐次微分方程的通解 $Y(x)$.

第二步，求该非齐次微分方程的特解 $y^*(x)$，命之如下：

$$y^*(x) = x^k Q_m(x) e^{ax}$$

其中 $Q_m(x)$ 为 x 的 m 次多项式，系数待定，

$$k = \begin{cases} 0, & \text{当 } a \text{ 不是特征根时} \\ 1, & \text{当 } a \text{ 为单重特征根时.} \\ 2, & \text{当 } a \text{ 为二重特征根时} \end{cases}$$

类型 2　方程 $y'' + py' + qy = P_m(x)e^{ax}\cos bx$ 或 $y'' + py' + qy = Q_m(x)e^{ax}\sin bx$ 或 $y'' + py' + qy = P_m(x)e^{ax}\cos bx + Q_m(x)e^{ax}\sin bx$ 的解法（其中 $P_m(x)$，$Q_m(x)$ 为 x 的 m 次已知多项式）：

第一步，按上面 1，写出对应的齐次微分方程的通解 $Y(x)$.

第二步，求该非齐次微分方程的特解 $y^*(x)$，命之如下：

$$y^*(x) = x^k (R_m(x)e^{ax}\cos bx + S_m(x)e^{ax}\sin bx)$$

其中 $R_m(x)$，$S_m(x)$ 为 x 的 m 次多项式，系数待定，

$$k = \begin{cases} 0, & \text{当 } a + ib \text{ 不是特征根时} \\ 1, & \text{当 } a + ib \text{ 为单重特征根时} \end{cases}$$