# Homework 5: Problems and Solutions

## Problem 1: Reversible kinetic-energy machines

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Consider a particle released from rest and lowered through a fixed vertical height $h$ by a *kinetic-energy machine*, as discussed in Lecture 8. Compare machines acting on the same particle between the same upper and lower levels. Treat the particle as a point mass, with no rotational energy. Each machine can operate repeatedly, returning to its initial mechanical state after each passage. It receives no external power and has no net change in stored energy; an irreversible machine may release heat to its surroundings.

A machine is *reversible* if reversing the particle's velocity at the lower level makes it retrace the process and return to rest at the upper level, restoring the machine to its original state.

Use the impossibility of a perpetual-motion machine to prove the following statements. Describe how the machines could be combined to obtain a contradiction if a statement were false.

**1.** No machine can give the particle a greater final speed than a reversible machine with the same vertical drop.

**2.** Any two reversible machines with the same vertical drop give the particle the same final speed, regardless of their design.

Base your argument on reversibility and the impossibility of perpetual motion; do not assume the mechanical-energy conservation law that this argument is intended to motivate.

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**Solution. 1. A reversible machine gives the greatest final speed.** Let $R$ be a reversible machine and suppose, for contradiction, that another machine $A$ gives the particle a greater final speed, $v_A > v_R$. After $A$ lowers the particle, reverse its velocity at the lower level and operate $R$ backward. Reversibility ensures that $R$ returns the particle to rest at the upper level and itself returns to its initial state. At the bottom, however, the particle has more kinetic energy than is needed to enter the reversed process of $R$. The excess is

$$\Delta K = \frac{1}{2} m(v_A^2 - v_R^2) > 0.$$

Thus the combined cycle restores the particle and both machines to their initial states while delivering positive energy. Repeating the cycle would produce energy without external input, a perpetual-motion machine. Hence $v_A \leq v_R$.

**2. All reversible machines give the same final speed.** Let two reversible machines give speeds $v_1$ and $v_2$. Applying part 1 with machine 1 as the candidate machine and machine 2 as

the reversible reference gives $v_1 \le v_2$. Interchanging their roles gives $v_2 \le v_1$. Therefore $v_1 = v_2$, independent of design.

Equivalently, if $v_1 > v_2$, one could use machine 1 for the downward passage, extract $\frac{1}{2}m(v_1^2 - v_2^2)$ at the bottom, and use machine 2 in reverse with the remaining speed $v_2$ to return the particle to rest at the top. The machines and particle would complete a cycle while positive energy was extracted, again contradicting the impossibility of perpetual motion.

## Problem 2: Energy of an interacting many-particle system

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A system consists of $N$ point particles with masses $m_i$, positions $\mathbf{r}_i$, and velocities $\mathbf{v}_i$. It is in a uniform gravitational field $-g\hat{\mathbf{z}}$, with $z$ measured vertically upward.

The interaction between particles $i$ and $j$ is a time-independent central force depending only on their separation $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$:

$$\mathbf{F}_{ij} = f_{ij}(r_{ij})\,\frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}}, \qquad f_{ij} = f_{ji},$$

where $\mathbf{F}_{ij}$ is the force on particle $i$ due to particle $j$. Thus $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$. There are no other forces.

**1.** Introduce a pair potential energy $U_{ij}(r_{ij})$ and state its relation to $f_{ij}(r_{ij})$. Write the conserved total energy of the system, including kinetic energy, gravitational potential energy, and the interaction potential energies. Explain why each interacting pair must be counted once. Give equivalent expressions for the interaction energy using $\sum_{i<j}$ and $\frac{1}{2}\sum_{i\neq j}$.

**2.** Starting from Newton's second law for each particle, show briefly that the time derivative of your total energy is zero. Show explicitly how the work done by the two forces in each interacting pair combines with the change in $U_{ij}$.

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**Solution. 1. Potential energy and total energy.** For each unordered pair $(i, j)$, introduce a pair potential $U_{ij}(r_{ij})$ such that

$$f_{ij}(r_{ij}) = -\frac{\mathrm{d}U_{ij}}{\mathrm{d}r_{ij}}.$$

Equivalently, the potential is obtained by integrating the force along the line of centers,

$$U_{ij}(r) = -\int_{r_0}^{r} f_{ij}(s)\,\mathrm{d}s,$$

which is the one-dimensional form of $U_{ij} = -\int \mathbf{F}_{ij}\cdot d\mathbf{r}'$: on that path $\mathbf{F}_{ij}$ is parallel to $d\mathbf{r}'$, so only the radial component contributes.

The total energy is

$$E = \sum_i \frac{1}{2}m_i v_i^2 + \sum_i m_i g z_i + \sum_{i<j} U_{ij}(r_{ij}).$$

Each interacting pair is counted once because $U_{ij}$ is the energy of the pair, not a separate energy assigned twice to its two members. Since $U_{ij} = U_{ji}$, the interaction energy can equivalently be written

$$E_{\mathrm{int}} = \sum_{i<j} U_{ij} = \frac{1}{2}\sum_{i\neq j} U_{ij}.$$

**2. Conservation of total energy.** Write the total energy of part 1 as $E = K + U_{\text{grav}} + U_{\text{int}}$, with

$$K = \sum_i \frac{1}{2} m_i v_i^2, \qquad U_{\text{grav}} = \sum_i m_i g z_i, \qquad U_{\text{int}} = \sum_{i<j} U_{ij}(r_{ij}).$$

Differentiating term by term gives the three contributions to $\mathrm{d}E/\,\mathrm{d}t$:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \underbrace{\sum_i m_i \mathbf{v}_i \cdot \dot{\mathbf{v}}_i}_{\mathrm{d}K/\,\mathrm{d}t} + \underbrace{\sum_i m_i g \dot{z}_i}_{\mathrm{d}U_{\text{grav}}/\,\mathrm{d}t} + \underbrace{\sum_{i<j} \dot{U}_{ij}}_{\mathrm{d}U_{\text{int}}/\,\mathrm{d}t}.$$

We evaluate the three in turn.

*(i) Kinetic term.* Newton's second law for particle $i$ reads

$$m_i \dot{\mathbf{v}}_i = -m_i g \hat{\mathbf{z}} + \sum_{j \neq i} \mathbf{F}_{ij}.$$

Taking the scalar product with $\mathbf{v}_i$ and summing over all particles gives

$$\frac{\mathrm{d}K}{\mathrm{d}t} = -\sum_i m_i g \dot{z}_i + \sum_i \sum_{j \neq i} \mathbf{v}_i \cdot \mathbf{F}_{ij}.$$

*(ii) Gravitational term.* Since $\mathrm{d}U_{\text{grav}}/\,\mathrm{d}t = \sum_i m_i g \dot{z}_i$, the first term of $\mathrm{d}K/\,\mathrm{d}t$ is exactly $-\,\mathrm{d}U_{\text{grav}}/\,\mathrm{d}t$. The gravitational contributions therefore cancel identically, leaving

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sum_i \sum_{j \neq i} \mathbf{v}_i \cdot \mathbf{F}_{ij} + \sum_{i<j} \dot{U}_{ij}.$$

It remains to show that these two terms cancel each other.

*(iii) Interaction term.* For one pair, combine the work rates of the two equal-and-opposite forces:

$$\mathbf{v}_i \cdot \mathbf{F}_{ij} + \mathbf{v}_j \cdot \mathbf{F}_{ji} = (\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{F}_{ij}.$$

Since $\mathbf{F}_{ij} = f_{ij} \hat{\mathbf{r}}_{ij}$, this equals $f_{ij}(\mathbf{v}_i - \mathbf{v}_j) \cdot \hat{\mathbf{r}}_{ij}$, and the last factor is exactly $\dot{r}_{ij}$: differentiating $r_{ij}^2 = (\mathbf{r}_i - \mathbf{r}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)$ with respect to time gives

$$2 r_{ij} \dot{r}_{ij} = 2(\mathbf{r}_i - \mathbf{r}_j) \cdot (\mathbf{v}_i - \mathbf{v}_j) \qquad \Longrightarrow \qquad \dot{r}_{ij} = \hat{\mathbf{r}}_{ij} \cdot (\mathbf{v}_i - \mathbf{v}_j).$$

On the other hand,

$$\dot{U}_{ij} = \frac{\mathrm{d}U_{ij}}{\mathrm{d}r_{ij}} \dot{r}_{ij} = -f_{ij} \dot{r}_{ij},$$

so the pair's work rate is minus the rate of change of its potential energy:

$$\mathbf{v}_i \cdot \mathbf{F}_{ij} + \mathbf{v}_j \cdot \mathbf{F}_{ji} = f_{ij} \dot{r}_{ij} = -\dot{U}_{ij}.$$

The double sum over ordered pairs must now be regrouped into unordered pairs, each of which then cancels one $\dot{U}_{ij}$:

$$\sum_i \sum_{j \neq i} \mathbf{v}_i \cdot \mathbf{F}_{ij} = \sum_{i<j} \big( \mathbf{v}_i \cdot \mathbf{F}_{ij} + \mathbf{v}_j \cdot \mathbf{F}_{ji} \big) = -\sum_{i<j} \dot{U}_{ij} = -\frac{\mathrm{d}U_{\text{int}}}{\mathrm{d}t}.$$

*Collecting the terms.* Substituting the three pieces into $\mathrm{d}E/\,\mathrm{d}t$ gives

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\mathrm{d}U_{\mathrm{grav}}}{\mathrm{d}t} + \frac{\mathrm{d}U_{\mathrm{int}}}{\mathrm{d}t} \\ &= \Big[-\sum_i m_i g \dot{z}_i + \sum_i \sum_{j\neq i} \mathbf{v}_i \cdot \mathbf{F}_{ij}\Big] + \sum_i m_i g \dot{z}_i + \sum_{i<j} \dot{U}_{ij} = 0.\end{aligned}$$

In the second line the two gravitational terms cancel, and the double sum equals $-\,\mathrm{d}U_{\mathrm{int}}/\,\mathrm{d}t$ after regrouping, so every term cancels and $\mathrm{d}E/\,\mathrm{d}t = 0$.

## Problem 3: A cube balanced on a cylinder

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A uniform cube of mass $m$ and edge length $2b$ rests on top of a fixed horizontal cylinder of radius $r$, as shown in Fig. 1. Initially, the cube's center lies vertically above the cylinder's axis. Consider small rocking motions in the plane of the figure, perpendicular to the cylinder's axis. Static friction is sufficient to prevent slipping, and the cube remains in contact with the cylinder along its bottom face. Take the gravitational acceleration to be $g$.

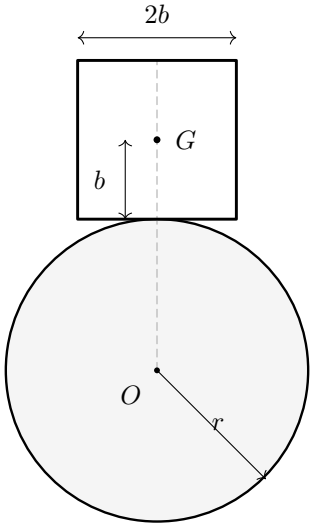



Fig. 1: The initial equilibrium. $O$ is the cylinder's center in cross section, and $G$ is the cube's center of mass.

Before calculating, use your intuition to predict whether increasing the cylinder's radius makes the equilibrium more or less stable.

**1.** Let the contact point move through a small angle $\theta$ from the top of the cylinder. Express the height $H(\theta)$ of the cube's center of mass above $O$. *Hint:* Without slipping, the distance rolled along the cube's bottom face is $r\theta$.

**2.** Expand $H(\theta) - H(0)$ to second order in $\theta$. Use the change in gravitational potential energy to find the condition for a positive quadratic term and hence a stable equilibrium. Compare with your initial prediction. The borderline case in which the quadratic term vanishes need not be analyzed.

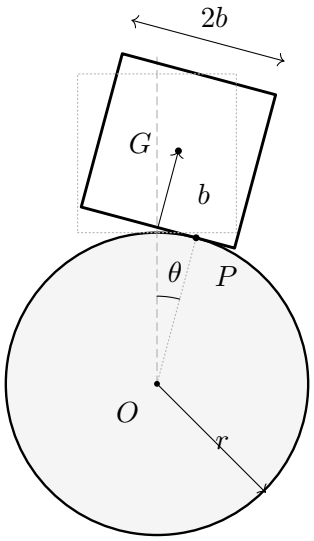
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**Solution.** Let $P$ be the instantaneous contact point, $C$ the center of the cube's bottom face, and $G$ its center of mass. Let $\theta$ be the angular displacement of the contact point from the top of the cylinder. The cube's bottom face is tangent at $P$, so its inclination is $\theta$. The no-slip condition gives the distance traveled along the face from $C$ to the contact location as $r\theta$. Taking the direction of motion to the right as positive, the vertical coordinates are

$$y_P = r\cos\theta, \qquad y_C - y_P = r\theta\sin\theta, \qquad y_G - y_C = b\cos\theta.$$

Consequently,

$$H(\theta) = (r+b)\cos\theta + r\theta\sin\theta.$$




For small $\theta$,

$$\cos\theta = 1 - \frac{\theta^2}{2} + O(\theta^4), \qquad \theta\sin\theta = \theta^2 + O(\theta^4),$$

and therefore

$$H(\theta) - H(0) = \frac{r-b}{2}\theta^2 + O(\theta^4).$$

The quadratic change in gravitational potential energy is $mg(r-b)\theta^2/2$. Thus the equilibrium is stable for $r > b$ and unstable for $r < b$. Increasing the cylinder radius makes the equilibrium more stable. The borderline case $r = b$ is not decided by the quadratic term.

## Problem 4: Tidal potential and equilibrium ocean height

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In Homework 4, the lunar tidal force was found to be, to first order in $|\mathbf{r}|/D$,

$$\mathbf{F}_{\text{tide}}(\mathbf{r}) = \frac{GMm}{D^3}\big[3\mathbf{n}(\mathbf{n}\cdot\mathbf{r}) - \mathbf{r}\big].$$

Here $M$ is the Moon's mass, $D$ is the Earth–Moon distance, $\mathbf{n}$ is a unit vector from Earth's center toward the Moon, and $\mathbf{r}$ is the position of a test mass $m$ relative to Earth's center. Use the same translating, nonrotating Earth-centered frame as in Homework 4.

**1.** Find a tidal potential energy $U_{\text{tide}}(\mathbf{r})$ satisfying $\mathbf{F}_{\text{tide}} = -\nabla U_{\text{tide}}$, $U_{\text{tide}}(\mathbf{0}) = 0$. Verify your result by taking its gradient.

**2.** Model Earth as a sphere of radius $R$, covered by an ocean in instantaneous equilibrium. Hold the Moon's direction fixed while finding the equilibrium surface. Neglect Earth's rotation, deformation of the solid Earth, and the ocean's self-gravity.

Let $h$ be the small sea-surface displacement from the reference sphere, with $|h| \ll R$, and take the terrestrial gravitational acceleration $g$ to be constant near the surface. The change in Earth's gravitational potential energy is then $mgh$. The free surface is an equipotential of the total potential energy.

To leading order in the tidal perturbation, evaluate $U_{\text{tide}}$ at $r = R\hat{\mathbf{r}}$ while retaining $mgh$ for the terrestrial contribution. Find the height difference

$$\Delta h = h_{\parallel} - h_{\perp},$$

where $h_{\parallel}$ is the surface height along the Earth–Moon line and $h_{\perp}$ is the height in a perpendicular direction. Express your answer in terms of $G$, $M$, $R$, $D$, and $g$, and identify which direction has the higher surface.

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**Solution. 1. Tidal potential.** Using $\nabla r^2 = 2\mathbf{r}$ and $\nabla(\mathbf{n}\cdot\mathbf{r})^2 = 2\mathbf{n}(\mathbf{n}\cdot\mathbf{r})$, a potential satisfying the specified force and $U_{\text{tide}}(\mathbf{0}) = 0$ is

$$U_{\text{tide}}(\mathbf{r}) = \frac{GMm}{2D^3}\left[r^2 - 3(\mathbf{n}\cdot\mathbf{r})^2\right].$$

Indeed, its negative gradient is $\frac{GMm}{D^3}[3\mathbf{n}(\mathbf{n}\cdot\mathbf{r}) - \mathbf{r}]$.

**2. Difference in equilibrium ocean height.** On the Earth–Moon line, $\mathbf{r} = R\mathbf{n}$, so $U_{\parallel} = -GMmR^2/D^3$. In a direction perpendicular to that line, $\mathbf{n}\cdot\mathbf{r} = 0$, so $U_{\perp} = GMmR^2/(2D^3)$. The free surface is an equipotential. Equating $mgh_{\parallel} + U_{\parallel}$ and $mgh_{\perp} + U_{\perp}$ gives

$$\Delta h = h_{\parallel} - h_{\perp} = \frac{U_{\perp} - U_{\parallel}}{mg} = \boxed{\frac{3GMR^2}{2gD^3}}.$$

This is positive, so the equilibrium surface is higher along the Earth–Moon line than in the perpendicular direction.