TA-Differential equation

Differential Equation

1. Physical laws $\rightarrow$ Relationships between rates of change (Newton's laws, circuit laws, etc.)
2. Equations for rates of change = differential equations
3. Solve differential equations = to predict the future of a system (position, voltage, temperature, probability, etc.)

First order Differential equation

Seperable equation

For the equation with the form

$$\int f(y)y'dx = \int g(x)dx$$
$$\int f(y)dy = \int f(y)\frac{dy}{dx}dx$$

$$\frac{dy}{dx} = xy$$

Solution

$$\frac{1}{y}dy = xdx$$
$$\int \frac{1}{y}dy = \int xdx$$
$$\ln |y| = \frac{x^2}{2} + C$$

The general solution to the equation

$$y = Ce^{x^2/2}$$

Homogeneous

The first order homogeneous equation have the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Usually, the equation is simplified by introducing the variable substitution $v = \frac{y}{x}$, after which it is transformed into a method that can separate variables to solve.

Example

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Solution

$$\frac{dy}{dx} = 1 + \frac{y}{x}$$

Set $v = \frac{y}{x}$, then $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$$
\begin{aligned}
v + x\frac{dv}{dx} &= 1 + v \\
x\frac{dv}{dx} &= 1
\end{aligned}
$$

Separate variables and integrate

$$
\begin{aligned}
\int dv &= \int \frac{1}{x} dx \\
v &= \ln|x| + C \\
\frac{y}{x} &= \ln|x| + C \\
y &= x(\ln|x| + C)
\end{aligned}
$$

First-order linear differential equation

$$y' + p(x)y = q(x)$$

we have a general solution

$$y = e^{-\int p(x)dx}\left[\int q(x)e^{\int p(x)dx}dx + C\right]$$

EXAMPLE ; KIRHOOFF RULES

Circuit: Resistor R in series with capacitor C, connected to a constant voltage source V_0.
According to Kirchhoff's laws:

$$V_0 = V_R + V_C = R\frac{dq}{dt} + \frac{q}{C}$$

Where is the charge on the capacitor. Equation from Kirchhoff's law (charge on the capacitor):

$$\frac{dq}{dt} + \frac{1}{RC_{\text{cap}}}q = \frac{V_0}{R}.$$

Here $p(t) = \frac{1}{RC_{\text{cap}}}$ and $q(t) = \frac{V_0}{R}$.
Integrating factor:

$$\mu(t) = \exp\left(\int \frac{1}{RC_{\text{cap}}}dt\right) = e^{t/(RC_{\text{cap}})}$$

Using the general formula:

$$q(t) = \frac{1}{\mu(t)}\left(\int \mu(t)\frac{V_0}{R}dt + C_1\right) = e^{-t/(RC_{\text{cap}})}\left(\frac{V_0}{R}\int e^{t/(RC_{\text{cap}})}dt + C_1\right).$$

Compute the integral:

$$\int e^{t/(RC_{\text{cap}})}dt = RC_{\text{cap}}\, e^{t/(RC_{\text{cap}})}$$

So

$$q(t) = e^{-t/(RC_{\text{cap}})}\left(\frac{V_0}{R}RC_{\text{cap}}\, e^{t/(RC_{\text{cap}})} + C_1\right) = V_0C_{\text{cap}} + C_1e^{-t/(RC_{\text{cap}})}.$$

Equation from Kirchhoff's law (charge $q(t)$ on the capacitor):

$$\frac{dq}{dt} + \frac{1}{RC_{\text{cap}}} q = \frac{V_0}{R}.$$

Here $P(t) = \frac{1}{RC_{\text{cap}}}$ and $Q(t) = \frac{V_0}{R}$, with integrating factor:

$$\mu(t) = \exp\left(\int \frac{1}{RC_{\text{cap}}} dt\right) = e^{t/(RC_{\text{cap}})}$$

Using the general formula:

$$q(t) = \frac{1}{\mu(t)}\left(\int \mu(t)\frac{V_0}{R} dt + C_1\right) = e^{-t/(RC_{\text{cap}})}\left(\frac{V_0}{R}\int e^{t/(RC_{\text{cap}})} dt + C_1\right).$$

Compute the integral:

$$\int e^{t/(RC_{\text{cap}})} dt = RC_{\text{cap}}\, e^{t/(RC_{\text{cap}})}$$

So

$$q(t) = e^{-t/(RC_{\text{cap}})}\left(\frac{V_0}{R} RC_{\text{cap}}\, e^{t/(RC_{\text{cap}})} + C_1\right) = V_0 C_{\text{cap}} + C_1 e^{-t/(RC_{\text{cap}})}.$$

Define the time constant $\tau = RC_{\text{cap}}$. With initial condition $q(0) = q_0$,

$$q_0 = V_0 C_{\text{cap}} + C_1 \Rightarrow C_1 = q_0 - V_0 C_{\text{cap}}.$$

Bernoulli's Equation

With the equation have the form

$$y' + p(x)y = q(x)y^n$$

convert to

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x)$$

Let

$$v = y^{1-n}, \quad \Rightarrow \quad \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Substitute into the equation:

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

Or, equivalently:

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

which is a first-order linear equation.

Example

$$\frac{dy}{dx} + y = y^2$$

Here, $P(x) = 1$, $Q(x) = 1$, and $n = 2$.

1. Rewrite the equation in standard form

$$\frac{dy}{dx} + y = y^2$$

2. Substitute $v = y^{1-2} = y^{-1}$.

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$$

3. The original equation:

$$\frac{dy}{dx} + y = y^2 \Rightarrow \frac{dy}{dx} = y^2 - y$$

Substitute:

$$\frac{dv}{dx} = -y^{-2}(y^2 - y) = -(1 - y^{-1}) = v - 1 \implies \frac{dv}{dx} - v = -1$$

4. This is a linear equation. The integrating factor is:

$$\mu(x) = e^{-\int 1dx} = e^{-x}$$

5. Solution:

$$\frac{d}{dx}(ve^{-x}) = -e^{-x}$$

Integrate:

$$ve^{-x} = \int -e^{-x}dx = e^{-x} + C$$
$$v = 1 + Ce^{x}$$

6. Substitute back $v = y^{-1}$:

$$\frac{1}{y} = 1 + Ce^{x} \implies y(x) = \frac{1}{1 + Ce^{x}}$$

second order

A second-order linear differential equation has the general form:

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

- If $g(x) = 0$, the equation is called a homogeneous second-order differential equation.
- If $g(x) \neq 0$, it is called a non-homogeneous equation.

homogeneous function

For the Constant Coefficient Second-Order Homogeneous Differential Equations have the form

$$ay'' + by' + cy = 0$$

Characteristic equation

$$ar^2 + br + c = 0$$

From the solution of the differential equation

- Two unequal real roots $r_1 \neq r_2$, the general solution is: $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
- Two equal real roots $r_1 = r_2 = r$, the general solution is: $y(x) = (C_1 + C_2 x) e^{rx}$
- Two conjugate complex roots $r_{1,2} = \alpha \pm \beta i$, the general solution is: $y(x) = e^{\alpha x} \left(C_1 \cos \beta x + C_2 \sin \beta x \right)$

Example : Solving a Second-Order Linear Homogeneous Equation with Constant Coefficients

Solving a Second-Order Linear Homogeneous Equation with Constant Coefficients

$$y'' - 3y' + 2y = 0$$

Characteristic equation

$$r^2 - 3r + 2 = 0$$

Solve the characteristic equation

$$(r-1)(r-2) = 0$$

The roots are $r_1 = 1$ and $r_2 = 2$.Write the general solution Since the characteristic equation has two distinct real roots, the general solution is

$$y(x) = C_1 e^x + C_2 e^{2x}$$

Example: Solving a Second-Order Linear Homogeneous Equation with Complex Roots

$$y'' + 4y = 0$$

Characteristic equation

$$r^2 + 4 = 0$$

Solve the characteristic equation

$$r = \pm 2i$$

Write the general solution Since the roots are complex, the general solution is

$$y(x) = C_1 \cos 2x + C_2 \sin 2x$$

Non-Homogeneous Term with Trigonometric Functions

$$y'' + 4y = \sin 2x$$

Find the general solution to the homogeneous equation
The corresponding homogeneous equation is

$$y'' + 4y = 0$$

The characteristic equation is

$$r^2 + 4 = 0$$

Solving, we get $r = \pm 2i$. So the general solution to the homogeneous equation is:

$$y_h(x) = C_1 \cos 2x + C_2 \sin 2x$$

Find the particular solution
The non-homogeneous term is $g(x) = \sin 2x$. We assume the particular solution has the form:

$$y_p(x) = A\cos 2x + B\sin 2x$$

Substituting this into the equation and solving gives $A = 0, B = \frac{1}{4}$. Therefore, the particular solution is:

$$y_p(x) = \frac{1}{4}\sin 2x$$

Write the general solution The total solution is

$$y(x) = C_1\cos 2x + C_2\sin 2x + \frac{1}{4}\sin 2x$$

non-homogeneous function

For non-homogeneous equations of the form $ay'' + by' + cy = g(x)$, the general solution is the sum of the solution to the homogeneous equation $y_h(x)$ and a particular solution $y_p(x)$

$$y(x) = y_h(x) + y_p(x)$$

where

- $y_h(x)$ is the **general solution** to the corresponding homogeneous equation, and
- $y_p(x)$ is a **particular solution** to the non-homogeneous equation.
  Methods such as variation of parameters and undetermined coefficients are commonly used to find the particular solution.

Example

Forced damping vibration we have the equation:

$$my'' + cy' + ky = F_0\cos(\omega t).$$

This is a second-order nonhomogeneous equation with constant coefficients.

- Convert to standard form:

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F_0}{m}\cos(\omega t).$$

To find the **Homogeneous solution** we solve the characteristic equation

$$r^2 + \frac{c}{m}r + \frac{k}{m} = 0$$

Three damping cases are obtained (underdamped, critically damped, and overdamped).

Then the **specific Solution**. The right side is a cosine function, a tentative solution.

$$y_p = A\cos(\omega t) + B\sin(\omega t).$$

Substituting A, B into the equation yields the steady-state response of the vibration.
we have **general solution**:

$$y(t) = y_h(t) + y_p(t)$$

The first half is the decaying "transient solution," while the second half is the stable "forced vibration."