

Mathematics Foundations

General Physics I Fall 2026

Tutorial #1 2026.9.12

Notes. Mathematics is a language for expressing a physical relationship precisely. Selected questions from the accompanying exercise sheet appear here as worked examples. First attempt them independently, then use the step-by-step solution to check the method.

Key idea. When solving a physics problem, first name the quantities and their units, then choose coordinates, write a symbolic relation, and only then substitute numbers. A result should pass a units check and a common-sense check.

A. Scalars and vectors

A.1 Two different kinds of quantity

A *scalar* is specified by one number (with a unit when physical): mass, time, temperature, energy, and distance are examples. A scalar is italic, for example m or t . A *vector* has both magnitude and direction: displacement, velocity, force, and angular momentum are examples. A vector is bold, for example $\mathbf{r}$ or $\mathbf{F}$. Unit vectors are denoted with a hat. Coordinate unit vectors are $\hat{x}$, $\hat{y}$, and $\hat{z}$. Scalars can be added to scalars of the same kind; vectors can be added only after their directions have been represented in a common coordinate system.

In Cartesian components,

$$\mathbf{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}, \quad |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}.$$

The components depend on the axes, but the geometric vector and its magnitude do not.

A.2 Vector arithmetic and products

Addition and scalar multiplication are component by component. Two especially useful products are

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = |\mathbf{a}| |\mathbf{b}| \cos \phi$$

and

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_y b_z - a_z b_y) \hat{x} + (a_z b_x - a_x b_z) \hat{y} \\ &\quad + (a_x b_y - a_y b_x) \hat{z}, \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \phi. \end{aligned}$$

The dot product is a scalar and measures alignment. The cross product is a vector, directed by the right-hand rule. In mechanics, they appear as work $W = \mathbf{F} \cdot \Delta \mathbf{r}$ and torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.

Example. Let $\mathbf{a} = (2, -1, 3)$ and $\mathbf{b} = (-1, 4, 2)$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{b}|$. What does the dot product tell you about the angle between the two vectors?

Solution. Add components for the vector sum, and multiply corresponding components for the dot product:

$$\mathbf{a} + \mathbf{b} = (2 - 1, -1 + 4, 3 + 2) = (1, 3, 5),$$

$$\mathbf{a} \cdot \mathbf{b} = 2 \times (-1) + (-1) \times 4 + 3 \times 2 = 0.$$

For the cross product, apply the component formula from A.2 one component at a time:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \hat{x}[(-1) \times 2 - 3 \times 4] + \hat{y}[3 \times (-1) - 2 \times 2] \\ &\quad + \hat{z}[2 \times 4 - (-1) \times (-1)] \\ &= -14\hat{x} - 7\hat{y} + 7\hat{z}.\end{aligned}$$

Finally,

$$|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}, \quad |\mathbf{b}| = \sqrt{(-1)^2 + 4^2 + 2^2} = \sqrt{21}.$$

Since $\mathbf{a} \cdot \mathbf{b} = 0$, the vectors are perpendicular.

Key idea. Before calculating, decide whether the desired answer should be a scalar or a vector. This makes it much easier to catch an incorrect dot product, cross product, or missing direction.

A.3 Cross products: two calculation methods and a formula sheet

The component formula above is the fastest way to calculate a cross product once it is familiar. The same calculation can be organized by the formal determinant

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

Expand along the first row: the middle cofactor has a minus sign. This immediately reproduces the component formula in A.2. The determinant is a mnemonic for the cross product, not an ordinary determinant whose entries are all numbers.

The following identities are useful in calculating. They hold for vectors in a right-handed three-dimensional Cartesian basis; no proofs are needed to use them.

Formula	Meaning or useful consequence
$\hat{x} \times \hat{y} = \hat{z}, \quad \hat{y} \times \hat{z} = \hat{x}, \quad \hat{z} \times \hat{x} = \hat{y}$	Cyclic order is positive; reversing an order changes its sign. Also $\hat{x} \times \hat{x} = 0$.
$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	The cross product is anti-commutative.
$(\alpha \mathbf{a} + \beta \mathbf{b}) \times \mathbf{c} = \alpha(\mathbf{a} \times \mathbf{c}) + \beta(\mathbf{b} \times \mathbf{c})$	Distribution law over addition and scalar multiplication.
$ \mathbf{a} \times \mathbf{b} = \mathbf{a} \mathbf{b} \sin \phi$	Magnitude equals the area of the parallelogram spanned by the two vectors.
$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$	The scalar triple product is cyclic; it is a signed volume.
$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$	Rules of vector cross product.
$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$	
$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\mathbf{d}$	

B. Matrices: calculations and transformations

B.1 Definition, addition, scalar multiplication, and transpose

An $m \times n$ matrix is a rectangular array with m rows and n columns:

$$A = (A_{ij})_{m \times n} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}.$$

Two matrices can be added only when they have the same dimensions. For matrices $A = (A_{ij})$ and $B = (B_{ij})$ of the same size and a scalar λ ,

$$(A + B)_{ij} = A_{ij} + B_{ij}, \quad (\lambda A)_{ij} = \lambda A_{ij}.$$

The transpose exchanges rows and columns:

$$(A^T)_{ij} = A_{ji}, \quad \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}.$$

Example. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Calculate A^T and $3B$.

Solution. First transpose A : its rows become columns. Then multiply every entry of B by 3:

$$A^T = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ -1 & 3 & 0 & 2 \end{pmatrix}, \quad 3B = \begin{pmatrix} 3 & 0 \\ 6 & 3 \\ 3 & 6 \end{pmatrix}.$$

B.2 Matrix multiplication

If A is $m \times n$ and B is $n \times p$, their product $C = AB$ exists and is an $m \times p$ matrix. Its entries are

$$C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}.$$

Read this as *row i of A dotted with column j of B* . The order matters: usually $AB \neq BA$.

A matrix can therefore transform a column vector. For example,

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \times x + 2 \times y \\ (-1) \times x + 3 \times y \end{pmatrix} = \begin{pmatrix} x + 2y \\ -x + 3y \end{pmatrix}.$$

Key idea. Before any matrix product, write down the dimensions. An $m \times n$ matrix can multiply an $n \times p$ matrix and produces an $m \times p$ matrix.

Example. Return to the matrices A and B above. Calculate AB .

Solution. Check dimensions first:

$$(4 \times 3)(3 \times 2) = 4 \times 2.$$

Thus each row of A multiplies both columns of B , producing eight row-by-column dot products:

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 & -1 \\ \mathbf{0} & \mathbf{1} & \mathbf{3} \\ 2 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{2} & 1 \\ \mathbf{1} & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 1 + 2 \times 2 + (-1) \times 1 & 1 \times 0 + 2 \times 1 + (-1) \times 2 \\ \mathbf{0} \times \mathbf{1} + \mathbf{1} \times \mathbf{2} + \mathbf{3} \times \mathbf{1} & 0 \times 0 + 1 \times 1 + 3 \times 2 \\ 2 \times 1 + (-1) \times 2 + 0 \times 1 & 2 \times 0 + (-1) \times 1 + 0 \times 2 \\ 1 \times 1 + 0 \times 2 + 2 \times 1 & 1 \times 0 + 0 \times 1 + 2 \times 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 5 & 7 \\ 0 & -1 \\ 3 & 4 \end{pmatrix} \end{aligned}$$

B.3 Determinants of square matrices

A determinant assigns one number to a square matrix. For a 2×2 matrix,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

For a 3×3 matrix, a direct calculation is the sum of the three downward diagonal products minus the sum of the three upward diagonal products:

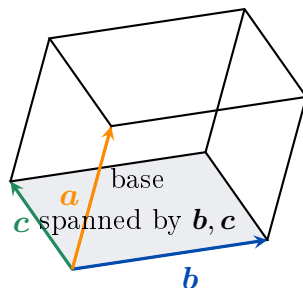
$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - bdi - afh.$$

The determinant is zero when a transformation squashes a volume (or, in two dimensions, an area) to zero. Its magnitude gives the scale factor for area or volume; its sign records whether orientation is reversed.

Example. The three edge vectors of a parallelepiped are $\mathbf{a} = (1, 2, -1)$, $\mathbf{b} = (3, 0, 4)$, and $\mathbf{c} = (2, -1, 1)$. Use the scalar triple product to find its volume.

Geometric picture. Take $\mathbf{b}$ and $\mathbf{c}$ as two adjacent base edges. The area of their parallelogram is $|\mathbf{b} \times \mathbf{c}|$. The dot product with $\mathbf{a}$ extracts the signed height component of the third edge, so

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$



Solution. The scalar triple product is a 3×3 determinant:

$$\begin{aligned} V &= \left| \det \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix} \right| \\ &= |1 \times 0 \times 1 + 2 \times 4 \times 2 + (-1) \times 3 \times (-1) \\ &\quad - (-1) \times 0 \times 2 - 2 \times 3 \times 1 - 1 \times 4 \times (-1)| \\ &= |0 + 16 + 3 - 0 - 6 + 4| \\ &= \boxed{17 \text{ units}^3}. \end{aligned}$$

C. Rotations and polar coordinates

C.1 Rotating a vector in the plane

An *active* counterclockwise rotation turns the vector while the axes remain fixed:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For a 90° counterclockwise rotation, (x, y) becomes $(-y, x)$. A rotation preserves a vector's length. If instead the axes rotate while the vector stays fixed, the coordinate components acquire the opposite sign of rotation; do not mix these two descriptions.

Example. (a) Write the active counterclockwise rotation matrix $R(30^\circ)$ in the plane. (b) Apply it to

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution. Start from the general active-rotation matrix and substitute $\cos 30^\circ = \sqrt{3}/2$ and $\sin 30^\circ = 1/2$:

$$R(30^\circ) = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}.$$

Multiply the matrix by the column vector one component at a time:

$$\begin{aligned} x' &= \frac{\sqrt{3}}{2} \times 2 + \left(-\frac{1}{2}\right) \times 1 = \sqrt{3} - \frac{1}{2}, \\ y' &= \frac{1}{2} \times 2 + \frac{\sqrt{3}}{2} \times 1 = 1 + \frac{\sqrt{3}}{2}. \end{aligned}$$

Answer. $R(30^\circ) = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$ and $R\mathbf{r} = \begin{pmatrix} \sqrt{3} - 1/2 \\ 1 + \sqrt{3}/2 \end{pmatrix}.$

C.2 Polar coordinates label position by distance and angle

For a point in a plane,

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}.$$

The angle θ is measured from the positive x axis. In physics, use *radians* in all formulas involving angles, especially before differentiating or using a small-angle approximation. One complete turn is 2π radians.

Example. (a) Convert $(x, y) = (-3, 3\sqrt{3})$ to polar coordinates (r, θ) with $0 \leq \theta < 2\pi$. (b) Convert $(r, \theta) = (4, 5\pi/6)$ to Cartesian coordinates.

Solution. For part (a), calculate the distance first:

$$r = \sqrt{(-3)^2 + (3\sqrt{3})^2} = \sqrt{9 + 27} = 6.$$

The point has $x < 0$ and $y > 0$, so it lies in quadrant II. Its reference angle is $\pi/3$ because $|y/x| = \sqrt{3}$. Therefore

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

For part (b), use the conversion formulas directly:

$$x = 4 \cos \frac{5\pi}{6} = -2\sqrt{3}, \quad y = 4 \sin \frac{5\pi}{6} = 2.$$

Key idea. A polar coordinate pair names a point; it does not introduce a new physical object. Draw the point before choosing the angle so that the quadrant is correct.

C.3 Velocity and acceleration in polar coordinates

Example. Work through the following guided problem.

- (a) Derive the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ in the polar basis $\hat{r}, \hat{\theta}$.
- (b) Specialize the result to circular motion of fixed radius $r = R$, allowing the angular speed to change with time.
- (c) Assume the particle moves in the $+\hat{\theta}$ direction and define its speed by $v = R\dot{\theta}$. Show that

$$\mathbf{a} = -\frac{v^2}{R}\hat{r} + \frac{dv}{dt}\hat{\theta}.$$

- (d) Which term changes the direction of the velocity, and which term changes its magnitude? What remains for uniform circular motion?

Solution. The polar unit vectors themselves rotate as the particle moves. Express them in the fixed Cartesian basis:

$$\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}, \quad \hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}.$$

Differentiate each one with respect to time:

$$\frac{d\hat{r}}{dt} = -\sin \theta \dot{\theta} \hat{x} + \cos \theta \dot{\theta} \hat{y} = \dot{\theta} \hat{\theta},$$

$$\frac{d\hat{\theta}}{dt} = -\cos \theta \dot{\theta} \hat{x} - \sin \theta \dot{\theta} \hat{y} = -\dot{\theta} \hat{r}.$$

Because the position vector is $\mathbf{r} = r\hat{r}$, the product rule gives

$$\begin{aligned}\mathbf{v} &= \frac{d}{dt}(r\hat{r}) \\ &= \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} \\ &= \boxed{\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}.\end{aligned}$$

Differentiate this expression once more. Keep the derivative of each unit vector:

$$\begin{aligned}\mathbf{a} &= \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \\ &= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} - r\dot{\theta}^2\hat{r} \\ &= \boxed{(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}.\end{aligned}$$

For circular motion, $r = R$ is constant, so $\dot{r} = \ddot{r} = 0$. The general formulas reduce to

$$\mathbf{v} = R\dot{\theta}\hat{\theta}, \quad \mathbf{a} = -R\dot{\theta}^2\hat{r} + R\ddot{\theta}\hat{\theta}.$$

For motion in the $+\hat{\theta}$ direction, italic $v = R\dot{\theta}$ is the speed, whereas bold $\mathbf{v}$ is the velocity vector. Since

$$R\dot{\theta}^2 = \frac{(R\dot{\theta})^2}{R} = \frac{v^2}{R}, \quad R\ddot{\theta} = \frac{d}{dt}(R\dot{\theta}) = \frac{dv}{dt},$$

we obtain

$$\mathbf{a} = -\frac{v^2}{R}\hat{r} + \frac{dv}{dt}\hat{\theta}.$$

Physical interpretation. In the general formula, $\ddot{r}\hat{r}$ changes the radial speed, $-r\dot{\theta}^2\hat{r}$ points inward, $r\ddot{\theta}\hat{\theta}$ comes from changing angular speed, and $2\dot{r}\dot{\theta}\hat{\theta}$ appears when radial motion occurs in a rotating polar basis. For circular motion, the inward component $-v^2\hat{r}/R$ is perpendicular to the velocity and changes its *direction*; the tangential component $(dv/dt)\hat{\theta}$ is parallel to the velocity and changes its *magnitude*. For uniform circular motion, $dv/dt = 0$, so only the centripetal acceleration remains.

D. Derivatives: rules and a function table

D.1 Rates of change and derivative rules

If $y = f(x)$, the same derivative may be written in any of the following forms:

$$\frac{dy}{dx} = f'(x) = y'.$$

All three denote the instantaneous rate of change with respect to x . Leibniz notation makes the differentiated variables explicit, while prime notation is compact. When the independent

variable is time, dot notation is standard in mechanics:

$$\dot{y} = \frac{dy}{dt}, \quad \ddot{y} = \frac{d^2y}{dt^2}.$$

D.2 Combining functions and differentiating an inverse

$$\frac{d}{dx}[f + g] = f' + g', \quad \frac{d}{dx}(fg) = f'g + fg', \quad \frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

The last identity is the chain rule. It explains the factor a in the derivatives of $\sin(ax)$ and e^{ax} . For a quotient, either use

$$\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{f'g - fg'}{g^2},$$

or rewrite it as $f(x)[g(x)]^{-1}$ and use the product and chain rules.

Inverse-function rule. If $y = f^{-1}(x)$, then $x = f(y)$. Differentiating $x = f(y)$ with respect to x gives $1 = f'(y) dy/dx$, so

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}.$$

An inverse function is not the reciprocal $1/f(x)$.

Example. Let $g(x) = e^{2x} \ln x + \arctan(x^2)$. Find $g'(x)$.

Solution. Separate the two terms. For the product $e^{2x} \ln x$, write

$$u = e^{2x}, \quad u' = 2e^{2x}, \quad v = \ln x, \quad v' = \frac{1}{x}.$$

The product rule gives

$$\frac{d}{dx}(e^{2x} \ln x) = u'v + uv' = 2e^{2x} \ln x + \frac{e^{2x}}{x}.$$

For the second term, suppose we do not yet know the derivative of $\arctan w$. Begin with $\tan u = \sin u / \cos u$. The quotient rule and the derivatives of sine and cosine give

$$\begin{aligned} \frac{d}{du} \tan u &= \frac{(\cos u)(\cos u) - (\sin u)(-\sin u)}{\cos^2 u} \\ &= \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = \sec^2 u. \end{aligned}$$

Now let $h(w) = \arctan w$, so $\tan[h(w)] = w$. Differentiate both sides with respect to w :

$$\sec^2[h(w)]h'(w) = 1.$$

Therefore, using $\sec^2 h = 1 + \tan^2 h$ and $\tan[h(w)] = w$,

$$h'(w) = \frac{1}{\sec^2[h(w)]} = \frac{1}{1 + w^2}.$$

Finally set $w = x^2$. The chain rule gives

$$\frac{d}{dx} \arctan(x^2) = h'(w) \frac{dw}{dx} = \frac{1}{1 + (x^2)^2} \times 2x = \frac{2x}{1 + x^4}.$$

Add the two derivatives:

$$g'(x) = e^{2x} \left(2 \ln x + \frac{1}{x} \right) + \frac{2x}{1 + x^4}.$$

Key idea. When a function has several layers, identify the outer function and the inner function before differentiating. The most common error is forgetting the derivative of the inner function.

D.3 Physics translation

For one-dimensional motion,

$$v(t) = \frac{dx}{dt}, \quad a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

For a vector-valued position, differentiate each component separately. The units change by one power of time in the denominator each time you differentiate.

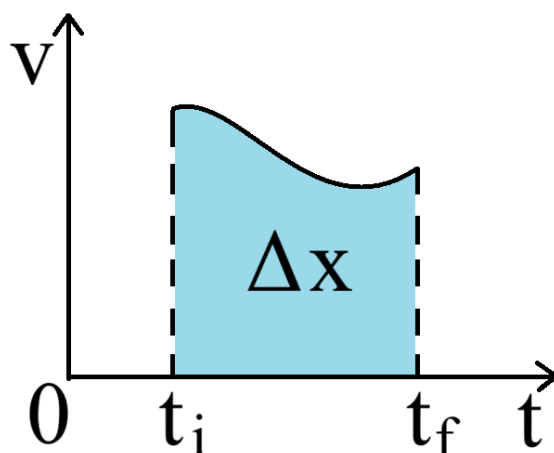
D.4 Derivative reference table

In the table, $b > 0$ and $b \neq 1$ are constants.

No.	$f(x)$	$f'(x)$	No.	$f(x)$	$f'(x)$
1	c	0	11	$\tan x$	$\sec^2 x$
2	x^n	nx^{n-1}	12	$\cot x$	$-\csc^2 x$
3	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	13	$\sec x$	$\sec x \tan x$
4	$1/x$	$-\frac{1}{x^2}$	14	$\csc x$	$-\csc x \cot x$
5	e^x	e^x	15	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
6	b^x	$b^x \ln b$	16	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
7	$\ln x $	$\frac{1}{x}$	17	$\arctan x$	$\frac{1}{1+x^2}$
8	$\log_b x $	$\frac{1}{x \ln b}$	18	$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$
9	$\sin x$	$\cos x$	19	$\operatorname{arcsec} x$	$\frac{1}{ x \sqrt{x^2-1}}$
10	$\cos x$	$-\sin x$	20	$\operatorname{arccsc} x$	$-\frac{1}{ x \sqrt{x^2-1}}$

E. Integrals and the Newton–Leibniz formula

E.1 From the *Area of a Curvilinear Trapezoid* to the Definite Integral



An integral adds many small contributions. For motion in one dimension, slice a velocity–time graph into narrow strips. Each strip has approximate area $v \Delta t$, which is a small displacement. Adding the strips gives the total signed displacement.

If the graph lies above the time axis, its area contributes positive displacement; a region below the axis contributes negative displacement. The same accumulation idea gives

$$\Delta x = \int_{t_i}^{t_f} v(t) dt, \quad \Delta v = \int_{t_i}^{t_f} a(t) dt, \quad W = \int_{x_i}^{x_f} F_x(x) dx.$$

E.2 Fundamental theorem of calculus

If $F'(x) = f(x)$, then

$$\boxed{\int_a^b f(x) dx = F(b) - F(a).}$$

This Newton–Leibniz formula connects an accumulated quantity to an antiderivative.

Example. A particle has velocity

$$v(t) = 3t^2 - 2t + 1$$

in m/s. Find its displacement from $t = 0$ to $t = 2$ s.

Solution. An antiderivative of $v(t)$ is

$$V(t) = t^3 - t^2 + t.$$

Therefore

$$\Delta x = \int_0^2 (3t^2 - 2t + 1) dt = [t^3 - t^2 + t]_0^2 = (8 - 4 + 2) - 0 = 6 \text{ m}.$$

E.3 Two techniques that reverse derivative rules

Substitution. If $u = g(x)$ and $du = g'(x) dx$, replace the whole inside function and its derivative:

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

For example, with $u = x^2$,

$$\int 2x \cos(x^2) dx = \int \cos u du = \sin(x^2) + C.$$

Integration by parts. Reverse the product rule:

$$\int u dv = uv - \int v du.$$

For example, taking $u = x$ and $dv = e^x dx$ gives

$$\int xe^x dx = xe^x - \int e^x dx = (x - 1)e^x + C.$$

Key idea. For a definite integral, substitute the upper limit and subtract the value at the lower limit. Under a substitution, either change the limits to the new variable or substitute back before applying the original limits.

E.4 Common antiderivatives: a reference table

The table collects common elementary integrals. Here $a > 0$ and $b > 0$, $b \neq 1$ are constants. Add $+C$ to every indefinite integral.

No.	$f(x)$	$\int f(x) dx$	No.	$f(x)$	$\int f(x) dx$
1	x^n ($n \neq -1$)	$\frac{x^{n+1}}{n+1} + C$	12	$\csc^2 x$	$-\cot x + C$
2	$\frac{1}{x}$	$\ln x + C$	13	$\sec x \tan x$	$\sec x + C$
3	e^x	$e^x + C$	14	$\csc x \cot x$	$-\csc x + C$
4	b^x	$\frac{b^x}{\ln b} + C$	15	$\sec x$	$\ln \sec x + \tan x + C$
5	$\ln x$	$x \ln x - x + C$	16	$\csc x$	$\ln \csc x - \cot x + C$
6	$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + C$	17	$\frac{1}{1+x^2}$	$\arctan x + C$
7	$\sin x$	$-\cos x + C$	18	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
8	$\cos x$	$\sin x + C$	19	$\frac{1}{a^2+x^2}$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
9	$\tan x$	$\ln \sec x + C$	20	$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) + C$
10	$\cot x$	$\ln \sin x + C$	21	$\frac{1}{x^2-a^2}$	$\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right + C$
11	$\sec^2 x$	$\tan x + C$	22	$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln \left \frac{a+x}{a-x} \right + C$

F. Taylor expansions: controlled approximations

F.1 The Taylor formula about any point

Near $x = a$, a smooth function can be approximated by its derivatives at a :

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

The more terms retained, the better the approximation is usually when x is close to a . The special case $a = 0$ is called a *Maclaurin series*.

F.2 Common Maclaurin series: a reference table

In this table, u must be dimensionless. The formulas for sine, cosine, and tangent use radians.

Function	Maclaurin series about $u = 0$
e^u	$1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \frac{u^6}{6!} + \dots$
$\sin u$	$u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!} - \dots$
$\cos u$	$1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \frac{u^8}{8!} - \dots$
$\tan u$	$u + \frac{u^3}{3} + \frac{2u^5}{15} + \frac{17u^7}{315} + \frac{62u^9}{2835} + \dots$
$\frac{1}{1-u}$	$1 + u + u^2 + u^3 + u^4 + u^5 + \dots$
$\frac{1}{1+u}$	$1 - u + u^2 - u^3 + u^4 - u^5 + \dots$
$\ln(1+u)$	$u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} - \frac{u^6}{6} + \dots$
$(1+u)^\alpha$	$1 + \alpha u + \frac{\alpha(\alpha-1)}{2!}u^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}u^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}u^4 + \dots$
$\sqrt{1+u}$	$1 + \frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{16} - \frac{5u^4}{128} + \frac{7u^5}{256} - \dots$
$\arctan u$	$u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \frac{u^9}{9} - \dots$

The exponential, sine, and cosine series converge for every finite u ; the tangent series requires $|u| < \pi/2$. The geometric, logarithmic, binomial, square-root, and arctangent series are normally used for $|u| < 1$; the endpoint details depend on the particular series.

F.3 A small parameter is an assumption

An approximation should always state what is small. For a pendulum of small angular displacement θ , the replacement $\sin \theta \approx \theta$ is a statement that $|\theta| \ll 1$ radian. It becomes less accurate as the angle grows.

Example. Write the Maclaurin expansion of e^x through the x^4 term and use it to approximate $e^{0.20}$.

Solution. The Maclaurin series for the exponential function is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Keep terms through x^4 and substitute $x = 0.20$ only after writing the series:

$$\begin{aligned} e^{0.20} &\approx 1 + 0.20 + \frac{(0.20)^2}{2} + \frac{(0.20)^3}{6} + \frac{(0.20)^4}{24} \\ &= 1 + 0.20 + 0.020000 + 0.001333 + 0.0000667 \\ &= 1.2214 \quad (\text{to four decimal places}). \end{aligned}$$

Key idea. Approximations are useful when you can estimate the size of the first omitted term. That estimate tells you whether the approximation is trustworthy.

G. First-order and second-order differential equations

G.1 Separation of variables: a general method

A first-order differential equation is *separable* when it can be written as

$$\frac{dy}{dx} = F(x)G(y),$$

where x is the independent variable and y is the unknown function. The key step is to put every factor involving y on one side and every factor involving x on the other:

$$\frac{dy}{G(y)} = F(x) dx.$$

If an initial condition $y(x_0) = y_0$ is known, integrate with limits already built in:

$$\boxed{\int_{y_0}^{y(x)} \frac{du}{G(u)} = \int_{x_0}^x F(s) ds.}$$

Here u and s are dummy variables of integration, so they do not conflict with the limits. Evaluate both integrals, then solve the resulting relation for $y(x)$ whenever possible. If y cannot be isolated, the implicit relation is still a valid solution.

This division assumes $G(y) \neq 0$. Therefore, also check separately whether any constant value satisfying $G(y) = 0$ is a solution; dividing by $G(y)$ can otherwise discard it.

Example. A velocity obeys

$$\frac{dv}{dt} = -\frac{v}{\tau}, \quad v(0) = v_0,$$

where the time constant τ is known. Find $v(t)$.

Solution. This has the general separable form with $x = t$, $y = v$, $F(t) = -1/\tau$, and $G(v) = v$. Explicitly,

$$\frac{dv}{v} = -\frac{dt}{\tau}.$$

Apply the initial condition as limits of integration:

$$\int_{v_0}^{v(t)} \frac{dv'}{v'} = -\int_0^t \frac{dt'}{\tau}.$$

Evaluating both integrals gives

$$\ln \frac{v(t)}{v_0} = -\frac{t}{\tau}.$$

Exponentiate both sides and multiply by v_0 :

$$\boxed{v(t) = v_0 e^{-t/\tau}.}$$

G.2 First-order linear differential equations

A general first-order linear differential equation has the form

$$y'(t) + p(t)y(t) = q(t).$$

To solve it, use an *integrating factor*. If $y(t_0) = y_0$, define

$$\mu(t) = \exp\left(\int_{t_0}^t p(s) ds\right).$$

Then calculate in this order:

$$\begin{aligned} y' + py &= q, \\ \mu y' + \mu py &= \mu q, \\ \frac{d}{dt} [\mu(t)y(t)] &= \mu(t)q(t), \\ \mu(t)y(t) - y_0 &= \int_{t_0}^t \mu(s)q(s) ds. \end{aligned}$$

Therefore

$$\boxed{y(t) = \frac{y_0 + \int_{t_0}^t \mu(s)q(s) ds}{\mu(t)}}.$$

For the special case $y' + ky = s$ with constant $k > 0$ and s , use $\mu = e^{k(t-t_0)}$ to obtain

$$y(t) = \frac{s}{k} + \left(y_0 - \frac{s}{k}\right) e^{-k(t-t_0)}.$$

The first term is the equilibrium value $y_{\text{eq}} = s/k$; the second term is the exponentially decaying difference from equilibrium. The units must satisfy $[q] = [y]/T$ and $[p] = T^{-1}$.

G.3 Constant-coefficient second-order homogeneous equations

A constant-coefficient second-order homogeneous differential equation has the form

$$a y'' + b y' + c y = 0, \quad a \neq 0.$$

Try an exponential solution $y = e^{rt}$. Substitution gives the *characteristic equation*

$$ar^2 + br + c = 0, \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The roots determine the calculation:

roots	general solution
$r_1 \neq r_2$ real	$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
$r_1 = r_2 = r$	$y = (C_1 + C_2 t) e^{rt}$
$\alpha \pm i\beta$	$y = e^{\alpha t} (C_1 \cos \beta t + C_2 \sin \beta t)$

For example, solve $y'' + 2y' + 5y = 0$ with $y(0) = 1$ and $y'(0) = 0$. The characteristic equation is

$$r^2 + 2r + 5 = 0,$$

so

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2} = -1 \pm 2i.$$

For roots $\alpha \pm i\beta = -1 \pm 2i$, the real general solution is

$$y(t) = e^{-t} (C_1 \cos 2t + C_2 \sin 2t).$$

Differentiate before substituting the initial conditions:

$$y'(t) = e^{-t} [(-C_1 + 2C_2) \cos 2t + (-C_2 - 2C_1) \sin 2t].$$

At $t = 0$,

$$y(0) = C_1 = 1, \quad y'(0) = -C_1 + 2C_2 = 0.$$

Thus $C_1 = 1$ and $C_2 = 1/2$, giving

$$y(t) = e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right).$$

The oscillation has angular frequency 2, while its amplitude decays with the envelope e^{-t} . The equation $mx'' + bx' + kx = 0$ for a damped mass-spring system is a physical example of this form.