

Exercises Sheet

Mathematics foundations for General Physics I

Instructions. During class, do it **yourself** without the help of the internet or AI. You can ask your peers or AI for help **after the lecture**. You do not need to worry about exercises marked with (★), which need some knowledge beyond this course's mathematical requirements.

A. Scalars and vectors

1. Let $\mathbf{a} = (2, -1, 3)$ and $\mathbf{b} = (-1, 4, 2)$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{b}|$. What does the dot product tell you about the angle between the two vectors?
2. A force $\mathbf{F} = (4, 1, -2)$ N acts at $\mathbf{r} = (2, -1, 3)$ m. Find the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, its magnitude, and its unit-vector direction. Verify that $\boldsymbol{\tau}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{F}$.

B. Matrices: calculations and transformations

1. Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}$. Calculate A^T , $3B$, and AB .
2. Let $P = \begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$. Calculate PQ and QP . Then find $\det P$, $\det Q$, and $\det(PQ)$, and verify that $\det(PQ) = (\det P)(\det Q)$.
3. The three edge vectors of a parallelepiped are $\mathbf{a} = (1, 2, -1)$, $\mathbf{b} = (3, 0, 4)$, and $\mathbf{c} = (2, -1, 1)$. Use the scalar triple product to find its volume.

C. Rotations and polar coordinates

1. (a) Write the active counterclockwise rotation matrix $R(30^\circ)$ in the plane. (b) Apply it to $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
2. Consider $R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. (a) What active three-dimensional rotation does this matrix represent?
(b★) What two conditions must a three-dimensional rotation matrix satisfy?
3. (a) Convert $(x, y) = (-3, 3\sqrt{3})$ to polar coordinates (r, θ) with $0 \leq \theta < 2\pi$. (b) Convert $(r, \theta) = (4, 5\pi/6)$ to Cartesian coordinates.
4. Work through the following guided problem. (a) Derive the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ in the polar basis $\hat{r}, \hat{\theta}$. (b) Specialize the result to circular motion of fixed radius $r = R$, allowing the angular speed to change with time. (c) Assume the particle moves in the $+\hat{\theta}$ direction and define its speed by $v = R\dot{\theta}$. Show that $\mathbf{a} = -v^2\hat{r}/R + (dv/dt)\hat{\theta}$. (d) Which term changes the direction of the velocity, and which term changes its magnitude? What remains for uniform circular motion?

D. Derivatives: rules and a function table

1. Let $g(x) = e^{2x} \ln x + \arctan(x^2)$. Find $g'(x)$.
2. A mass on a spring has $x(t) = A \cos(\omega t + \phi)$. Differentiate twice and show that it obeys $a = -\omega^2 x$. State the dimensions of ω .
3. Let $\frac{5}{4}y - 2 \sin \frac{(y+1)\pi}{6} = x^x$. Find $\left. \frac{dy}{dx} \right|_{x=2}$.

E. Integrals and the Newton–Leibniz formula

1. A particle has velocity $v(t) = 3t^2 - 2t + 1$ with units m/s. Find its displacement from $t = 0$ to $t = 2$ s.
2. Evaluate $\int_0^1 2x \cos(x^2) dx$.
3. Evaluate $\int_0^1 x^2 \ln(1+x) dx$.

F. Taylor expansions: controlled approximations

1. Write the Maclaurin expansion of e^x through the x^4 term and use it to approximate $e^{0.20}$.
2. Evaluate $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$ using Taylor expansions.
3. Use a Taylor approximation to fill in the blank to three decimal places:

A Japanese man came into the restaurant. I had seen him before, wandering around; he was trying to sell abacuses. He started to talk to the waiters and challenged them: he said he could add numbers faster than any of them could do. . . . Cube roots! He wants to do cube roots by arithmetic. It's hard to find a more difficult fundamental problem in arithmetic. It must have been his top-notch exercise in abacus-land.

He writes down a number on some paper — any old number — and I still remember it: 1729.03. He starts working on it, mumbling and grumbling: “Mmmmmmagmmmmbrrr” — he's working like a demon! He's poring away, doing this cube root.

Meanwhile I'm just sitting there. One of the waiters says, “What are you doing?” I point to my head. “Thinking!” I say. I write down 12 on the paper. After a little while I've got _____.

The man with the abacus wipes the sweat off his forehead: “Twelve!” he says. . . . The waiters are all excited and happy. They tell the man, “Look! He does it only by thinking, and you need an abacus! He's got more digits!”

— Adapted from *Surely You're Joking, Mr. Feynman!*

4. (★) Let $0 < a < e$. A horizontal line $y = b$ intersects the graph of $f(x) = \frac{ae}{2}x^2 - (a+e)x + \ln x$ at three positive abscissas $x_1 < x_2 < x_3$. Prove that

$$\frac{2}{e} + \frac{e-a}{6e^2} < x_1 + x_3 < \frac{2}{a} - \frac{e-a}{6e^2}.$$

G. First-order and second-order differential equations

1. A velocity obeys $dv/dt = -v/\tau$, with $v(t=0) = v_0$ and τ known. Find $v(t)$.
2. Solve the first-order linear equation $y' + 2y = 6$ with $y(0) = 1$ by using an integrating factor. State the equilibrium value and calculate $y(1)$.
3. Solve $y'' + 4y' + 3y = 0$ with $y(0) = 2$ and $y'(0) = -1$. Show the characteristic equation, find the two constants, and identify the long-time behavior.

Answers and solutions

Quick answers.

A1. $(1, 3, 5)$, 0 , $(-14, -7, 7)$, $\sqrt{14}$, $\sqrt{21}$; perpendicular.

A2. $(-1, 16, 6) \text{ N} \cdot \text{m}$; $|\boldsymbol{\tau}| = \sqrt{293} \text{ N} \cdot \text{m}$; direction $(-1, 16, 6)/\sqrt{293}$.

B1. $A^T = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 0 \\ -1 & 3 & 0 & 2 \end{pmatrix}$, $3B = \begin{pmatrix} 3 & 0 \\ 6 & 3 \\ 3 & 6 \end{pmatrix}$,

$AB = \begin{pmatrix} 4 & 0 \\ 5 & 7 \\ 0 & -1 \\ 3 & 4 \end{pmatrix}$.

B2. $PQ = \begin{pmatrix} 5 & -4 \\ 9 & 6 \end{pmatrix}$, $QP = \begin{pmatrix} 8 & -7 \\ 6 & 3 \end{pmatrix}$; determinants 11, 6, 66.

B3. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 17$; volume 17 units³.

C1. $R = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$; $R\mathbf{r} = \begin{pmatrix} \sqrt{3} - \frac{1}{2} \\ 1 + \frac{\sqrt{3}}{2} \end{pmatrix}$.

C2. $+90^\circ$ about the $+z$ -axis; $R^T R = I$, $\det R = 1$.

C3. $(6, 2\pi/3)$; $(-2\sqrt{3}, 2)$.

C4. (a) $\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$; $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$. **(b)**

For $r = R$, $\mathbf{v} = R\dot{\theta}\hat{\theta}$ and $\mathbf{a} = -R\dot{\theta}^2\hat{r} + R\ddot{\theta}\hat{\theta}$. **(d)** The radial term changes direction; the tangential term changes speed. Uniform circular motion leaves only inward centripetal acceleration.

D1. $g' = e^{2x}(2\ln x + 1/x) + 2x/(1+x^4)$.

D2. $a = -\omega^2 x$; $[\omega] = \text{s}^{-1}$.

D3. $\frac{48(1 + \ln 2)}{15 + 2\pi\sqrt{3}}$.

E1. 6 m.

E2. $\sin 1$.

E3. $\frac{2}{3} \ln 2 - \frac{5}{18}$.

F1. $e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \dots$; $e^{0.20} \approx 1.2214$.

F2. $1/2$.

F3. 12.002.

G1. $v(t) = v_0 e^{-t/\tau}$.

G2. $y = 3 - 2e^{-2t}$; equilibrium 3; $y(1) = 3 - 2e^{-2}$.

G3. $y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$; $y \rightarrow 0$.

A. Scalars and vectors

2. Use $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$:

$$\boldsymbol{\tau} = (-1, 16, 6) \text{ N} \cdot \text{m}, \quad |\boldsymbol{\tau}| = \sqrt{(-1)^2 + 16^2 + 6^2} = \sqrt{293} \text{ N} \cdot \text{m}.$$

Its unit-vector direction is $(-\hat{x} + 16\hat{y} + 6\hat{z})/\sqrt{293}$. Direct calculation gives $\boldsymbol{\tau} \cdot \mathbf{r} = 0$ and $\boldsymbol{\tau} \cdot \mathbf{F} = 0$.

B. Matrices: calculations and transformations

2. Ordinary matrix multiplication gives

$$PQ = \begin{pmatrix} 5 & -4 \\ 9 & 6 \end{pmatrix}, \quad QP = \begin{pmatrix} 8 & -7 \\ 6 & 3 \end{pmatrix}.$$

Also, $\det P = 2 \times 4 - 1 \times (-3) = 11$ and $\det Q = 1 \times 0 - (-2) \times 3 = 6$. Therefore $\det(PQ) = 5 \times 6 - (-4) \times 9 = 66 = 11 \times 6$.

C. Rotations and polar coordinates

2. The xy block sends (x, y) to $(-y, x)$, which is a 90° counterclockwise rotation. The z component is unchanged, so this is a rotation about the positive z -axis. Any proper three-dimensional rotation matrix satisfies $R^T R = I$ and $\det R = +1$.

D. Derivatives: rules and a function table

2. Differentiating gives $v = -A\omega \sin(\omega t + \phi)$ and $a = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x$. Since ωt must be dimensionless, $[\omega] = \text{s}^{-1}$.

3. The right-hand side is not a power rule alone. First rewrite it as

$$x^x = e^{x \ln x}.$$

The chain rule then gives

$$\frac{d}{dx} x^x = e^{x \ln x} \frac{d}{dx} (x \ln x) = x^x (\ln x + 1).$$

For the sine term, the chain rule gives

$$\frac{d}{dx} \left[-2 \sin \frac{(y+1)\pi}{6} \right] = -\frac{\pi}{3} \cos \frac{(y+1)\pi}{6} y'.$$

Thus implicit differentiation gives

$$\left(\frac{5}{4} - \frac{\pi}{3} \cos \frac{(y+1)\pi}{6} \right) y' = x^x (\ln x + 1).$$

At $x = 2$, $y = 4$ satisfies the original equation because $\frac{5}{4}(4) - 2 \sin(5\pi/6) = 5 - 1 = 4 = 2^2$. Therefore

$$y' = \frac{4(1 + \ln 2)}{5/4 + \pi\sqrt{3}/6} = \boxed{\frac{48(1 + \ln 2)}{15 + 2\pi\sqrt{3}}}.$$

E. Integrals and the Newton–Leibniz formula

2. Let $u = x^2$, so $du = 2x dx$. The limits remain 0 and 1, and

$$\int_0^1 2x \cos(x^2) dx = \int_0^1 \cos u du = \sin 1.$$

3. Integration by parts with $u = \ln(1+x)$ and $dv = x^2 dx$ gives

$$\int_0^1 x^2 \ln(1+x) dx = \frac{1}{3} \ln 2 - \frac{1}{3} \int_0^1 \frac{x^3}{1+x} dx.$$

Since $x^3/(1+x) = x^2 - x + 1 - 1/(1+x)$, the remaining integral is $5/6 - \ln 2$. Thus

$$\boxed{\int_0^1 x^2 \ln(1+x) dx = \frac{2}{3} \ln 2 - \frac{5}{18}}.$$

F. Taylor expansions: controlled approximations

2. Use $\tan x = x + x^3/3 + O(x^5)$ and $\sin x = x - x^3/6 + O(x^5)$. Then

$$\frac{\tan x - \sin x}{x^3} = \frac{x^3(1/3 + 1/6) + O(x^5)}{x^3} = \frac{1}{2} + O(x^2) \longrightarrow \boxed{\frac{1}{2}}.$$

3. Since $1729.03 = 1728 + 1.03 = 12^3 + 1.03$, write

$$\sqrt[3]{1729.03} = 12(1 + u)^{1/3}, \quad u = \frac{1.03}{1728}.$$

The Taylor expansion to the first order gives $(1 + u)^{1/3} \approx 1 + u/3$, thus

$$\sqrt[3]{1729.03} \approx 12 \left(1 + \frac{1.03}{3 \times 1728} \right) = 12 + \frac{4.12}{1728} = 12.002384 \dots$$

Therefore the entry in the blank, to three decimal places, is 12.002.

Supplementary material for Padé approximations

Let $f(x)$ have a power series about $x = 0$. Its $[m, n]$ Padé approximant is a rational function

$$R_{m,n}(x) = \frac{P_m(x)}{Q_n(x)}, \quad Q_n(0) = 1,$$

where P_m and Q_n have degrees at most m and n . The coefficients are chosen so that

$$Q_n(x)f(x) - P_m(x) = O(x^{m+n+1}).$$

Thus the Taylor series of $R_{m,n}$ agrees with that of f through degree $m + n$. The condition $Q_n(0) = 1$ removes the arbitrary common scale of the numerator and denominator.

For

$$L(x) = \ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5),$$

write the $[2, 2]$ candidate as

$$R_{2,2}(x) = \frac{p_1x + p_2x^2}{1 + q_1x + q_2x^2}.$$

The numerator has no constant term because $L(0) = 0$. Multiplying instead of dividing makes the coefficient matching transparent:

$$\begin{aligned} & (1 + q_1x + q_2x^2)L(x) - (p_1x + p_2x^2) \\ &= (1 - p_1)x + \left(-\frac{1}{2} + q_1 - p_2\right)x^2 + \left(\frac{1}{3} - \frac{q_1}{2} + q_2\right)x^3 \\ & \quad + \left(-\frac{1}{4} + \frac{q_1}{3} - \frac{q_2}{2}\right)x^4 + O(x^5). \end{aligned}$$

Set the coefficients of x, x^2, x^3, x^4 equal to zero. In order,

$$p_1 = 1, \quad p_2 = q_1 - \frac{1}{2}, \quad q_2 = \frac{q_1}{2} - \frac{1}{3}, \quad -\frac{1}{4} + \frac{q_1}{3} - \frac{q_2}{2} = 0.$$

Substitute the third equation into the fourth: $-1/12 + q_1/12 = 0$. Hence

$$q_1 = 1, \quad q_2 = \frac{1}{6}, \quad p_2 = \frac{1}{2},$$

and therefore

$$\boxed{R_{2,2}(x) = \frac{x + x^2/2}{1 + x + x^2/6}}, \quad \ln(1 + x) - R_{2,2}(x) = O(x^5).$$

Some low-order Padé approximants of $\ln(1+x)$ are collected below. Each is constructed about $x=0$.

order	$R_{m,n}(x)$	$\ln(1+x) - R_{m,n}(x)$
[1,0]	x	$O(x^2)$
[1,1]	$\frac{x}{1+x/2}$	$O(x^3)$
[1,2]	$\frac{x}{1+x/2-x^2/12}$	$O(x^4)$
[2,0]	$x - \frac{x^2}{2}$	$O(x^3)$
[2,1]	$\frac{x+x^2/6}{1+2x/3}$	$O(x^4)$
[2,2]	$\frac{x+x^2/2}{1+x+x^2/6}$	$O(x^5)$
[3,0]	$x - \frac{x^2}{2} + \frac{x^3}{3}$	$O(x^4)$

These are local approximations, and an approximant must not be evaluated at a zero of its denominator.

A Padé lower bound used in Exercise F4. To approximate $\ln x$ near $x=1$, substitute $x-1$ into the [2,2] formula:

$$R_{2,2}(x-1) = \frac{(x-1) + (x-1)^2/2}{1 + (x-1) + (x-1)^2/6} = \frac{3(x^2-1)}{x^2+4x+1}.$$

This rational function is in fact a strict lower bound for every $x > 1$. Define

$$D(x) = \ln x - \frac{3(x^2-1)}{x^2+4x+1}.$$

Then $D(1) = 0$, and direct differentiation and simplification give

$$D'(x) = \frac{1}{x} - \frac{12(x^2+x+1)}{(x^2+4x+1)^2} = \frac{(x-1)^4}{x(x^2+4x+1)^2} > 0 \quad (x > 1).$$

Consequently,

$$\boxed{\ln x > \frac{3(x^2-1)}{x^2+4x+1} \quad (x > 1).}$$

This boxed inequality is the precise point at which the [2,2] Padé approximant enters the following proof.

4. (Adapted from the 2022 Zhejiang Gaokao mathematics examination.)

Step 1: locate the three intersections. Differentiate:

$$f'(x) = aex - (a+e) + \frac{1}{x} = \frac{(ax-1)(ex-1)}{x}.$$

Because $0 < a < e$, the two positive critical points are $1/e < 1/a$. For a horizontal line to cut the graph three times, the intersections must lie on the three monotonic branches:

$$x_1 < \frac{1}{e} < x_2 < \frac{1}{a} < x_3.$$

Step 2: obtain an equation for $S = x_1 + x_3$. Set

$$S = x_1 + x_3, \quad r = \frac{x_3}{x_1}.$$

The ordering above gives

$$r > \frac{1/a}{1/e} = \frac{e}{a} > 1.$$

Since $f(x_3) = f(x_1) = b$, subtract the two equalities:

$$\frac{ae}{2}(x_3^2 - x_1^2) - (a + e)(x_3 - x_1) + \ln \frac{x_3}{x_1} = 0.$$

Divide by $x_3 - x_1 > 0$. Because

$$x_3 - x_1 = S \frac{r - 1}{r + 1}, \quad \frac{\ln r}{x_3 - x_1} = \frac{1}{S} \frac{r + 1}{r - 1} \ln r,$$

we obtain, after multiplying by $2S/(ae)$,

$$\boxed{S^2 - \left(\frac{2}{e} + \frac{2}{a}\right)S + \frac{2}{ae}h(r) = 0}, \quad h(r) = \frac{r + 1}{r - 1} \ln r.$$

Step 3: compare $h(r)$ with $h(e/a)$. For $r > 1$,

$$h'(r) = \frac{r - r^{-1} - 2 \ln r}{(r - 1)^2}.$$

Let $N(r) = r - r^{-1} - 2 \ln r$. Then

$$N(1) = 0, \quad N'(r) = 1 + \frac{1}{r^2} - \frac{2}{r} = \frac{(r - 1)^2}{r^2} > 0.$$

Thus $N(r) > 0$ and $h'(r) > 0$ for $r > 1$. Put

$$k = \frac{e}{a} > 1.$$

Since $r > k$, monotonicity gives $h(r) > h(k)$.

Step 4: use the $[2, 2]$ Padé lower bound. Apply the boxed Padé inequality to $x = k$:

$$\ln k > \frac{3(k^2 - 1)}{k^2 + 4k + 1}.$$

Therefore

$$\begin{aligned} h(r) &> h(k) = \frac{k + 1}{k - 1} \ln k \\ &> \frac{k + 1}{k - 1} \frac{3(k^2 - 1)}{k^2 + 4k + 1} = \frac{3(k + 1)^2}{k^2 + 4k + 1}. \end{aligned}$$

If $z = (k - 1)/(k + 1)$, then

$$\frac{3(k + 1)^2}{k^2 + 4k + 1} = \frac{2}{1 - z^2/3} > 2 \left(1 + \frac{z^2}{3}\right),$$

where the last inequality follows from $1/(1 - q) > 1 + q$ for $0 < q < 1$.

Step 5: convert the bound into the required interval. Let

$$L = \frac{2}{e} + \frac{e - a}{6e^2}, \quad U = \frac{2}{a} - \frac{e - a}{6e^2}.$$

First,

$$L + U = \frac{2}{e} + \frac{2}{a}, \quad U - L = \frac{(e - a)(6e - a)}{3ae^2} > 0.$$

Since $a = e/k$, Step 4 gives

$$\frac{2h(r)}{ae} > \frac{4k}{e^2} \left(1 + \frac{(k-1)^2}{3(k+1)^2} \right).$$

A direct common-denominator calculation yields

$$\begin{aligned} & \frac{4k}{e^2} \left(1 + \frac{(k-1)^2}{3(k+1)^2} \right) - LU \\ &= \frac{(k-1)^2(36k^3 - 23k^2 - 10k + 1)}{36e^2k^2(k+1)^2} > 0. \end{aligned}$$

Indeed,

$$36k^3 - 23k^2 - 10k + 1 = 4 + 52(k-1) + 85(k-1)^2 + 36(k-1)^3 > 0.$$

Hence $2h(r)/(ae) > LU$. Using the boxed quadratic equation from Step 2,

$$\begin{aligned} (S-L)(S-U) &= S^2 - (L+U)S + LU \\ &= LU - \frac{2h(r)}{ae} < 0. \end{aligned}$$

Because $L < U$, this is equivalent to $L < S < U$, i.e.

$$\boxed{\frac{2}{e} + \frac{e-a}{6e^2} < x_1 + x_3 < \frac{2}{a} - \frac{e-a}{6e^2}}.$$

G. First-order and second-order differential equations

2. The integrating factor is $\mu(t) = e^{2t}$. Therefore

$$(e^{2t}y)' = 6e^{2t}, \quad e^{2t}y = 3e^{2t} + C.$$

So $y = 3 + Ce^{-2t}$. The initial condition gives $C = -2$, hence $y = 3 - 2e^{-2t}$, with equilibrium value 3 and $y(1) = 3 - 2e^{-2}$.

3. The characteristic equation is $m^2 + 4m + 3 = 0$, with roots $m = -1, -3$. Hence

$$y = C_1e^{-t} + C_2e^{-3t}.$$

The initial conditions give $C_1 + C_2 = 2$ and $-C_1 - 3C_2 = -1$, so $C_1 = 5/2$ and $C_2 = -1/2$. Therefore

$$\boxed{y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}},$$

which approaches 0 as $t \rightarrow \infty$.