

Supplementary Reading

Dark matter: from orbital motion to resonance

Supplement to L6 (resonance) and L7 (Kepler's law)

Orbital motion lets us infer gravitating mass. Resonance lets us search for an extremely weak interaction in the laboratory. Together they connect a familiar mechanics question, “What produces this motion?”, to a modern one: “What is dark matter made of?”

E.1 Weighing a galaxy with orbital motion

For a nearly circular orbit around a dominant central mass M , Newton's law gives

$$\frac{mv_c^2}{r} = \frac{GMm}{r^2} \implies v_c = \sqrt{\frac{GM}{r}}, \quad T^2 = \frac{4\pi^2 r^3}{GM}.$$

Thus the Keplerian prediction is $v_c \propto r^{-1/2}$: farther out, the orbital speed decreases. A galaxy distributes its mass over a large volume. In a spherical approximation, replace M by the mass $M(< r)$ enclosed within r :

$$M(< r) = \frac{rv_c^2(r)}{G}.$$

An approximately flat rotation curve (旋转曲线), $v_c(r) \simeq v_f$, then implies $M(< r) \propto r$ over that radial interval. The inferred mass continues to grow where the starlight is already faint.^[1]

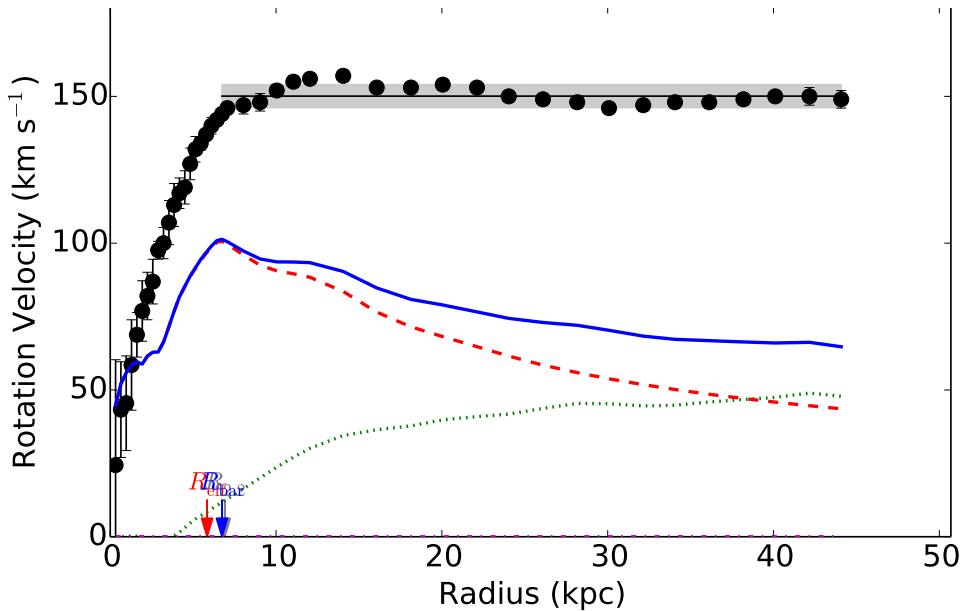


Fig. 1: NGC 3198, original SPARC rotation-curve panel. Black points: observed speeds; blue: the adopted stars-plus-gas prediction; red dashed: stellar disk; green dotted: gas. The outer speeds remain near 150 km s^{-1} . The blue curve includes an extended disk, rather than a point-mass approximation.^[1]

What is actually inferred?

At $r = 30 \text{ kpc}$ and $v_c \simeq 150 \text{ km s}^{-1}$, the spherical estimate is

$$M(< 30 \text{ kpc}) \simeq 1.6 \times 10^{11} M_{\odot}, \quad 1 \text{ kpc} \simeq 3.09 \times 10^{19} \text{ m}.$$

This estimates total gravitating mass, including ordinary matter. Stellar light and gas observations provide an independent estimate of the baryonic (重子物质) contribution. Within Newtonian gravity, their shortfall motivates an extended dark-matter halo (暗物质晕). Disk geometry, distance, inclination and the stellar mass-to-light ratio enter quantitative modeling; the spherical formula is a teaching approximation. Rotation curves constrain gravity and mass distribution, but do not identify a dark-matter particle.^[1]

E.2 Listening for an oscillating dark-matter field

Suppose dark matter is a sufficiently light boson, such as an axion (轴子), with a large occupation number. It can behave locally as a classical oscillating field. Its leading frequency is set by its rest energy:

$$f_{\text{DM}} \simeq \frac{m_{\text{DM}} c^2}{h}.$$

For example, $m_{\text{DM}} c^2 = 2 \text{ neV}$ corresponds to $f_{\text{DM}} \simeq 484 \text{ kHz}$. This is a field-oscillation frequency, distinct from the orbital frequency of matter in the Galaxy.^[2]

If an axion couples to electromagnetism, a static magnetic field can enable a tiny alternating electromagnetic signal. DMRadio uses resonant circuits and superconducting readout to search for such weak signals.^[3]

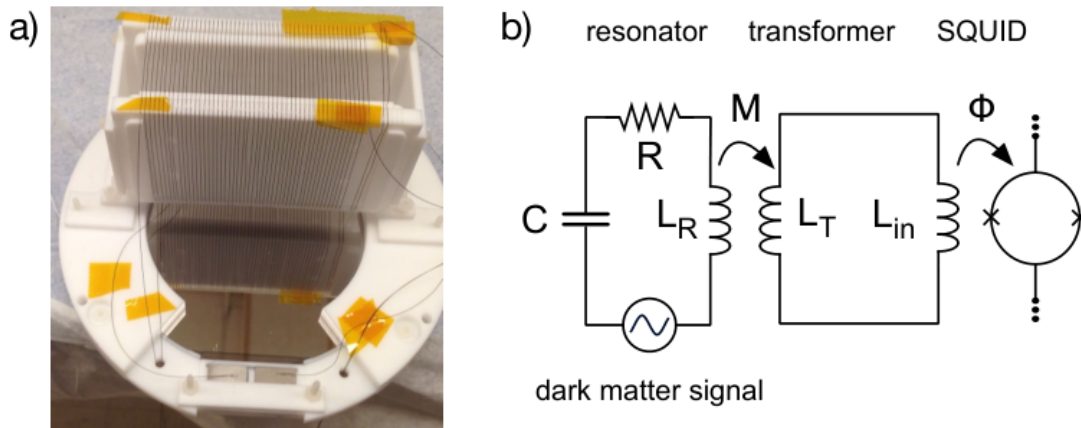


Fig. 2: An early fixed-frequency DMRadio prototype and its equivalent circuit, from Phipps et al. This apparatus searched for dark photons (暗光子) through kinetic mixing. The axion implementation additionally uses an applied static magnetic field. SQUID: superconducting quantum interference device (超导量子干涉仪).^[2]

The oscillator equation becomes a detector

For a simple series- RLC model, let q be capacitor charge and $I = \dot{q}$. An effective sinusoidal signal voltage ε_{DM} drives

$$\underbrace{L\ddot{q}}_{\text{inertia}} + \underbrace{R\dot{q}}_{\text{damping}} + \underbrace{\frac{q}{C}}_{\text{restoring term}} = \varepsilon_{\text{DM}} \cos \omega t.$$

The inductance L plays the role of mass; R supplies damping; $1/C$ plays the role of spring stiffness. For the steady response, write $q(t) = \text{Re}(\tilde{q}e^{i\omega t})$:

$$\tilde{q} = \frac{\varepsilon_{\text{DM}}}{C^{-1} - L\omega^2 + iR\omega}, \quad f_0 = \frac{1}{2\pi\sqrt{LC}}.$$

Changing C tunes f_0 through possible dark-matter frequencies. At $\omega_0 = 2\pi f_0$, the reactive terms cancel and the current amplitude is $\varepsilon_{\text{DM}}/R$. For weak damping, the quality factor (品质因数) and power-response linewidth satisfy

$$Q_{\text{res}} = \frac{\omega_0 L}{R}, \quad \Delta f_{\text{res}} \simeq \frac{f_0}{Q_{\text{res}}}.$$

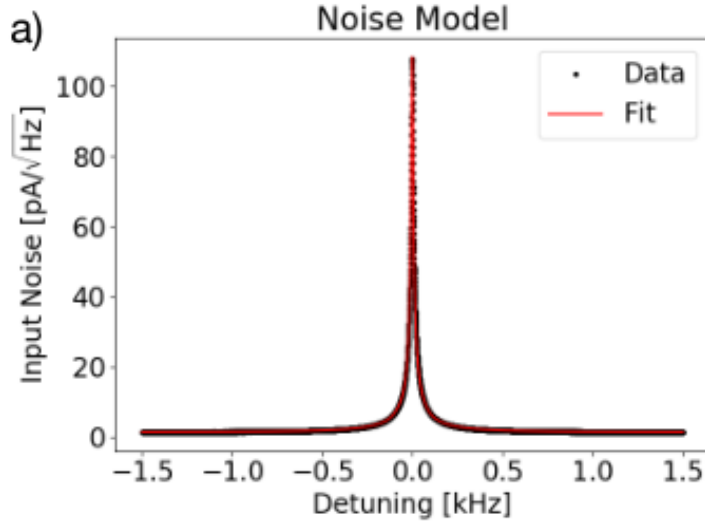


Fig. 3: Measured thermal-noise resonance of the early prototype: $f_0 = 492.027$ kHz and $Q_{\text{res}} \simeq 40,000$. The horizontal axis is detuning from f_0 ; the vertical axis is current-noise amplitude spectral density (幅度谱密度) in pA/ $\sqrt{\text{Hz}}$. This peak characterizes noise and resonator response; it is not a dark-matter detection.^[2]

The corresponding power-response full width at half maximum (半高全宽) is about 12.3 Hz. Its width must be extracted from the power-response model; the plotted amplitude spectral density is the square root of a power spectral density.

E.3 A resonance can also reveal a particle

Changing the momentum of a K^- beam incident on protons reveals peaks in the cross sections (反应截面) for two different final states:

$$K^- + p \longrightarrow \Lambda + \pi^+ + \pi^-,$$

$$K^- + p \longrightarrow \bar{K}^0 + n.$$

Here K^- is a negatively charged kaon (K 介子) and Λ is a hyperon (超子). Peaks at a common collision energy suggest an intermediate state accessible to both channels:

$$K^- + p \longrightarrow \mathcal{R} \longrightarrow \text{different final states.}$$

This is the particle-physics example in the resonance chapter: a short-lived state can reveal itself through an enhanced reaction probability. The original plot scans incident momentum, which determines the center-of-mass energy for a stationary proton target.^[4]

From a frequency peak to an energy peak

For $\ddot{x} + \gamma\dot{x} + \omega_0^2x = (F_0/m)\cos\omega t$, weak damping gives, near resonance,

$$|\tilde{x}(\omega)|^2 \propto \frac{1}{(\omega - \omega_0)^2 + (\gamma/2)^2}.$$

Near a narrow, isolated particle resonance, the analogous Breit–Wigner (布赖特–维格纳) shape is

$$\sigma_{\text{res}}(E) \propto \frac{1}{(E - E_R)^2 + (\Gamma/2)^2}, \quad E = \text{center-of-mass energy.}$$

This approximation takes the width and other factors as slowly varying across the peak and neglects interference with a nonresonant background. The peak energy is approximately $E_R = Mc^2$, where M is the resonance mass; Γ is its intrinsic energy full width at half maximum.

The frequency–energy relation $E = \hbar\omega$ connects the two line shapes. An exponentially decaying state has amplitude

$$a(t) \propto e^{-iE_R t/\hbar} e^{-\Gamma t/(2\hbar)}, \quad |a(t)|^2 \propto e^{-t/\tau}, \quad \boxed{\tau = \frac{\hbar}{\Gamma}}.$$

A narrower intrinsic peak therefore corresponds to a longer lifetime. Experimental resolution and backgrounds must be accounted for when extracting Γ .

What is resonating? In DMRadio, the detector provides the tunable resonance. In this scattering example, the peak reveals an unstable state of the interacting particles. The same line-shape mathematics describes different physical systems; a hadronic resonance is not itself evidence for dark matter.

E.4 Other searches for an oscillating field

Different couplings make different detectors useful. Compare these experiments with the lumped-element (集总参数) circuit of DMRadio:

Experiment	Physical response	Connection to resonance
ADMX ^[5]	Axion–photon coupling in a static magnetic field could generate microwave power.	Tune an electromagnetic cavity mode (微波腔模) to the axion frequency; read out with low-noise electronics.
CASPEr ^[6]	Axion/ALP interactions could drive precession of polarized nuclear spins.	Tune the Larmor frequency (拉莫尔频率) with a static magnetic field; detect transverse magnetization.
ABRACADABRA (10 cm) ^[7]	Axion–photon coupling in a toroidal magnet could generate an oscillating magnetic flux in its central region.	The cited 2021 search used broadband SQUID readout, providing a comparison with narrow-band resonant searches.

For CASPEr the tuning condition is

$$\omega_L = |\gamma_N|B_0 \simeq \omega_{\text{DM}},$$

where γ_N is the nuclear gyromagnetic ratio (旋磁比), distinct from the damping rate γ above. Its spin couplings differ from the photon coupling targeted by ADMX. A broadband readout can cover many frequencies simultaneously; a tuned resonator trades bandwidth for enhanced response. Comparing sensitivity also requires specifying noise, coupling and observation time.

E.5 A narrow resonance is only the beginning

The Galactic velocity distribution gives the dark-matter field a spread in kinetic energy. For typical speeds $v \sim 10^{-3}c$,

$$\frac{\Delta f_{\text{DM}}}{f_{\text{DM}}} \sim 10^{-6}, \quad t_{\text{coh}} \sim \frac{1}{\Delta f_{\text{DM}}},$$

as order-of-magnitude estimates; precise factors depend on the coherence-time convention and velocity distribution. The field has finite coherence time (相干时间) rather than a phase that stays fixed forever.^[2, 3]

A resonator enhances both a signal and some noise. Sensitivity therefore depends on losses, temperature, readout noise, integration time and the scan strategy. Increasing Q_{res} narrows the instantaneous bandwidth; the benefit must be evaluated together with the signal linewidth and the time required to cover an unknown frequency.^[3]

From evidence to identification. Orbital motion constrains the gravitational mass distribution. A resonant search tests a specified field, coupling and mass range. A null result limits that model under stated assumptions about the local dark-matter density; it leaves other candidates and untested couplings open. A candidate excess needs follow-up checks, including environmental backgrounds, retuning and the expected dependence on experimental parameters.

References

- [1] F. Lelli, S. S. McGaugh and J. M. Schombert, *SPARC: Mass Models for 175 Disk Galaxies with Spitzer Photometry and Accurate Rotation Curves*, *AJ* **152**, 157 (2016). Fig. 1 uses their NGC 3198 mass-model panel (stellar $M/L = 0.5$ in solar units at $3.6\,\mu\text{m}$); original kinematic sources are listed in the SPARC catalog.
[Paper: arXiv:1606.09251](#); [SPARC data and original figures](#).
- [2] A. Phipps et al., *Exclusion limits on hidden-photon dark matter near 2 neV from a fixed-frequency superconducting lumped-element resonator* (2019). Figs. 2 and 3 reproduce their Figs. 1 and 2(a).
[arXiv:1906.08814](#).
- [3] DMRadio Collaboration, *Projected sensitivity of DMRadio- m^3 : A search for the QCD axion below $1\,\mu\text{eV}$* , *Phys. Rev. D* **106**, 103008 (2022). This is a design/sensitivity study, not a detection claim.
[arXiv:2204.13781](#); [Stanford Physics: DMRadio 50 Liter](#).
- [4] R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. I, Secs. 23–2 and 23–4, especially Fig. 23–11. The scattering data are credited there to Ferro-Luzzi et al., *Phys. Rev. Lett.* **8**, 28 (1962). The constant-width line shape and lifetime relation above give the modern interpretation.
[Chapter 23 and original reaction-cross-section plots](#).
- [5] ADMX Collaboration, *The Axion Dark Matter eXperiment*, official experiment overview.
[University of Washington: ADMX](#).
- [6] CASPER Collaboration, official research overview: nuclear-spin precession, gradient and electric-dipole couplings.
[Mainz Budker Group: CASPER](#).
- [7] C. P. Salemi et al., *The search for low-mass axion dark matter with ABRACADABRA-10cm*, *Phys. Rev. Lett.* **127**, 081801 (2021). The cited run used a broadband configuration.
[arXiv:2102.06722](#).