Lecture Notes on Forced Oscillators Using Complex Exponentials

1 Introduction

In the study of forced oscillations, using complex exponentials can simplify the mathematical derivations and provide deeper insights into the behavior of the system. In this lecture, we will re-derive the key equations governing forced oscillators using the complex exponential form $e^{i\omega t}$. This approach leverages Euler's formula and simplifies the handling of sinusoidal functions.

2 The Forced Oscillator Equation

Consider a mass m attached to a spring with spring constant k, subject to a damping force proportional to its velocity (damping coefficient b), and driven by an external periodic force. The equation of motion is:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos(\omega t) \tag{1}$$

To facilitate the use of complex exponentials, we express the driving force as:

$$F(t) = \mathrm{Re}\left\{F_0 e^{i\omega t}\right\} \tag{2}$$

We will solve the equation using $F_0 e^{i\omega t}$ and take the real part of the solution at the end.

3 Solution of the Equation

The general solution $x(t)$ consists of:

(1) **Homogeneous Solution (x_h)**: Solution to the homogeneous equation (when $F_0 = 0$).

(2) **Particular Solution (x_p)**: A specific solution that accounts for the driving force.

Thus:

$$x(t) = x_h(t) + x_p(t) \tag{3}$$

3.1 Homogeneous Solution

The homogeneous equation is:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{4}$$

Divide through by m:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \tag{5}$$

Define:

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{(natural frequency)} \tag{6}$$

$$2\beta = \frac{b}{m} \quad \text{(damping coefficient)} \tag{7}$$

The equation becomes:

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = 0 \tag{8}$$

The characteristic equation is:

$$r^2 + 2\beta r + \omega_0^2 = 0 \tag{9}$$

Solving for r:

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{10}$$

Underdamped Case ($\beta^2 < \omega_0^2$)

The roots are complex:

$$r = -\beta \pm i\omega_1 \tag{11}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

The homogeneous solution is:

$$x_h(t) = e^{-\beta t}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) \tag{12}$$

This represents oscillations with exponentially decreasing amplitude.

3.2 Particular Solution Using Complex Exponentials

We propose a particular solution of the form:

$$x_p(t) = \tilde{X}e^{i\omega t} \tag{13}$$

where $\tilde{X}$ is a complex amplitude to be determined.

We substitute $x_p(t)$ into the non-homogeneous equation:

$$m\frac{d^2x_p}{dt^2} + b\frac{dx_p}{dt} + kx_p = F_0 e^{i\omega t} \tag{14}$$

Compute the derivatives:

$$\frac{dx_p}{dt} = i\omega \tilde{X} e^{i\omega t} \tag{15}$$

$$\frac{d^2x_p}{dt^2} = -\omega^2 \tilde{X} e^{i\omega t} \tag{16}$$

Substitute back into the equation:

$$\left(-m\omega^2 \tilde{X} + ib\omega \tilde{X} + k\tilde{X}\right) e^{i\omega t} = F_0 e^{i\omega t} \tag{17}$$

Divide both sides by $e^{i\omega t}$:

$$\left(-m\omega^2 + ib\omega + k\right) \tilde{X} = F_0 \tag{18}$$

Solve for $\tilde{X}$

Express the left side in terms of ω_0 and β:

$$-m\omega^2 + ib\omega + k = m\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right) \tag{19}$$

So:

$$m\left(\omega_0^2 - \omega^2 + 2i\beta\omega\right) \tilde{X} = F_0 \tag{20}$$

Then:

$$\tilde{X} = \frac{F_0}{m\left(\omega_0^2 - \omega^2 + 2i\beta\omega\right)} \tag{21}$$

Compute the Amplitude A and Phase ϕ

The physical solution is the real part of $x_p(t)$:

$$x_p(t) = \mathrm{Re}\left\{\tilde{X} e^{i\omega t}\right\} \tag{22}$$

Express $\tilde{X}$ in polar form:
Let:

$$D(\omega) = \omega_0^2 - \omega^2 + 2i\beta\omega \tag{23}$$

Compute the magnitude and phase of $D(\omega)$:
1. **Magnitude**:

$$|D(\omega)| = \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (2\beta\omega)^2} \tag{24}$$

So:

$$|\tilde{X}| = \frac{F_0}{m|D(\omega)|} \tag{25}$$

2. **Phase**:

$$\phi = \arg\left(D(\omega)\right) = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \tag{26}$$

Since $\tilde{X} = \dfrac{F_0}{mD(\omega)}$, the phase of $\tilde{X}$ is the negative of the phase of $D(\omega)$:

$$\tilde{X} = \frac{F_0}{m|D(\omega)|}e^{-i\phi} \tag{27}$$

Final Expression for $x_p(t)$

Thus, the particular solution is:

$$x_p(t) = \mathrm{Re}\left\{\tilde{X}e^{i\omega t}\right\} = |\tilde{X}|\cos(\omega t - \phi) \tag{28}$$

Therefore, the amplitude and phase angle are:

- **Amplitude**:

$$A = |\tilde{X}| = \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \left(2\beta\omega\right)^2}} \tag{29}$$

- **Phase Angle**:

$$\phi = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \tag{30}$$

This matches the results obtained using sine and cosine functions.

4 Resonance

Resonance occurs when the driving frequency ω is close to the natural frequency ω_0.

4.1 Amplitude at Resonance

At resonance ($\omega = \omega_0$):

$$|D(\omega_0)| = |2i\beta\omega_0| = 2\beta\omega_0 \tag{31}$$

The amplitude becomes:

$$A_{\max} = \frac{F_0/m}{2\beta\omega_0} = \frac{F_0}{2m\beta\omega_0} \tag{32}$$

This shows that the maximum amplitude is inversely proportional to the damping coefficient β.

4.2 Phase at Resonance

At resonance:

$$\phi = \arctan\left(\frac{2\beta\omega_0}{0}\right) = \frac{\pi}{2} \tag{33}$$

The oscillator lags the driving force by 90°.

5 Phase Relationships

- **Low Frequencies** ($\omega \ll \omega_0$):

$$\phi \approx \arctan(0) = 0 \tag{34}$$

The oscillator is in phase with the driving force.

- **At Resonance** ($\omega = \omega_0$):

$$\phi = \frac{\pi}{2} \tag{35}$$

- **High Frequencies** ($\omega \gg \omega_0$):

$$\phi \approx \arctan\left(\frac{2\beta\omega}{-\omega^2}\right) \approx \arctan(-0) = \pi \tag{36}$$

The oscillator is out of phase with the driving force.

6 Quality Factor and Damping

The **Quality Factor** Q quantifies the sharpness of the resonance peak:

$$Q = \frac{\omega_0}{2\beta} = \frac{m\omega_0}{b} \tag{37}$$

A higher Q indicates a lower rate of energy loss relative to the stored energy.

6.1 Bandwidth

The **bandwidth** $\Delta\omega$ is defined as the range of frequencies over which the power is greater than half its maximum value. For small damping:

$$\Delta\omega = \frac{\omega_0}{Q} = 2\beta \tag{38}$$

This shows that the bandwidth is inversely proportional to Q.

7 Energy Considerations

The average power supplied by the driving force equals the average power dissipated by damping.

7.1 Average Power Supplied

The instantaneous power is:

$$P(t) = F(t) \cdot v(t) = F_0 \cos(\omega t) \cdot v(t) \tag{39}$$

The velocity is:

$$v(t) = \frac{dx_p}{dt} = -A\omega \sin(\omega t - \phi) \tag{40}$$

So the instantaneous power is:

$$P(t) = -F_0 A\omega \cos(\omega t) \sin(\omega t - \phi) \tag{41}$$

Using trigonometric identities, the average power over one period $T = \dfrac{2\pi}{\omega}$ is:

$$\langle P \rangle = \frac{1}{2} F_0 A\omega \sin(\phi) \tag{42}$$

At resonance ($\phi = \dfrac{\pi}{2}$):

$$\langle P_{\max} \rangle = \frac{1}{2} F_0 A_{\max} \omega_0 \tag{43}$$

Substituting $A_{\max}$:

$$\langle P_{\max} \rangle = \frac{F_0^2}{4m\beta} \tag{44}$$

8 Examples of Forced Oscillators

1. **Mechanical Systems**: Bridges and buildings can resonate due to external forces like wind or earthquakes.
2. **Electrical Circuits**: An RLC circuit driven by an AC source behaves similarly to a mechanical oscillator.
3. **Optical Systems**: Resonance in optical cavities enhances certain frequencies of light.
4. **Biological Systems**: The human ear utilizes resonance to amplify specific sound frequencies.

9 Practical Applications

Resonance is utilized in various technologies:

- Radio tuners select desired frequencies.
- Quartz watches use mechanical resonance for timekeeping.
- Microwave ovens exploit molecular resonance of water.

- Pushing a child on a swing at the right moment to increase amplitude.
- Breaking a glass with a sound at its resonant frequency.

10 Example: Dark Matter Field Coupling Through B-L

In modern physics, it's hypothesized that dark matter may interact with normal matter through the B-L (Baryon-minus-Lepton) quantum number[1]. Suppose the external force acting on the oscillator is due to a dark matter field $\phi(t)$ coupling to the B-L charge Q_{B-L}:

$$F(t) = gQ_{B-L}\phi(t), \tag{45}$$

where g is the coupling constant.

Assuming $\phi(t) = \phi_0 \cos(\omega_{\mathrm{DM}} t)$, the force becomes:

$$F(t) = gQ_{B-L}\phi_0 \cos(\omega_{\mathrm{DM}} t). \tag{46}$$

This introduces an external driving force at frequency ω_{DM}, which can resonate with the oscillator under certain conditions, enhancing the sensitivity to dark matter detection.

10.1 Response of the Oscillator

Using the previous results, the amplitude of the oscillation due to the dark matter force is:

$$X_{\mathrm{DM}} = \frac{gQ_{B-L}\phi_0}{\sqrt{(k - m\omega_{\mathrm{DM}}^2)^2 + (b\omega_{\mathrm{DM}})^2}}. \tag{47}$$

At resonance ($\omega_{\mathrm{DM}} \approx \omega_0$ and low damping), the amplitude can become significantly large, making the detection of such a force feasible.

10.2 Coding Task Introduction

Our goal is to model the forced harmonic oscillator system to evaluate the sensitivity of an experimental setup to dark matter. By coding the equations of motion, we can simulate the response of the system to different dark matter parameters and optimize the design for maximum sensitivity.

10.2.1 Objectives

- Implement the differential equation of the forced harmonic oscillator in a computational tool (e.g., Python, MATLAB).
- Simulate the system's response to an external force due to a dark matter field.
- Analyze how changes in parameters (m, b, k, g, ω_{DM}) affect the amplitude and phase of the oscillation.
- Explore resonance conditions and damping effects.

10.2.2 Expected Outcomes

By completing this coding task, you will:

- Gain a deeper understanding of forced oscillations and resonance phenomena.
- Develop computational skills in modeling physical systems.
- Contribute to the exploration of dark matter detection methods using mechanical oscillators.

References

[1] Daniel Carney, Anson Hook, Zhen Liu, Jacob M. Taylor, Yue Zhao, "Ultralight dark matter detection with mechanical quantum sensors", *New J. Phys.*, vol. 23, no. 2, 2021, pp. 023041, doi: 10.1088/1367-2630/abd9e7.