Lecture 14: Euler Equation

14.1 Euler's equations

Space axes: nonrotating x, y, z directions fixed relative to an inertial frame.

Body frame: the rotating principal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$, with moments of inertia $\lambda_1, \lambda_2, \lambda_3$, respectively.

Choose O at a point fixed in an inertial frame, where the support force has zero torque about O, or at the center of mass with positions and velocities measured relative to it. In either case, the space axes remain nonrotating and $(d\vec{L}/dt)_{\text{space}} = \vec{\Gamma}$ about O; center-of-mass translation is treated separately.

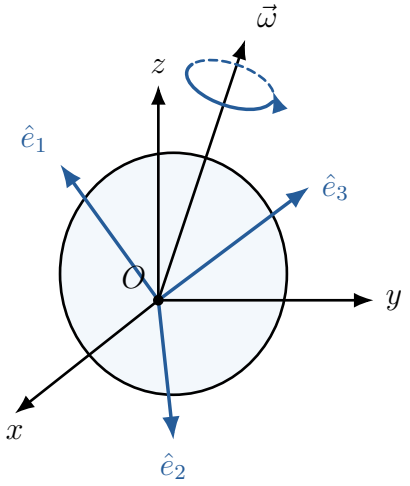


Fig. 1: The same motion described using fixed space axes and rotating body axes.

If the rigid body rotates with angular velocity $\vec{\omega}$, we can write in the body frame

$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3,$$

and

$$\vec{L} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3.$$

The torque is $\vec{\Gamma}$:

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{space}} = \vec{\Gamma} = \Gamma_1 \hat{e}_1 + \Gamma_2 \hat{e}_2 + \Gamma_3 \hat{e}_3 \,.$$

$$\boxed{\left(\frac{d\hat{e}_i}{dt}\right)_{\text{space}} = \vec{\omega} \times \hat{e}_i}$$

In the body principal axes, the inertia components are constant. The basis vectors rotate in space, so differentiating $\vec{L}$ gives an additional term $\vec{\omega} \times \vec{L}$:

$$\begin{aligned}\left(\frac{d\vec{L}}{dt}\right)_{\text{space}} &= \sum_{i=1}^{3} \lambda_i \dot{\omega}_i \hat{e}_i + \sum_{i=1}^{3} \lambda_i \omega_i \left(\frac{d\hat{e}_i}{dt}\right)_{\text{space}} \\ &= \lambda_1 \dot{\omega}_1 \hat{e}_1 + \lambda_2 \dot{\omega}_2 \hat{e}_2 + \lambda_3 \dot{\omega}_3 \hat{e}_3 \\ &\quad + \vec{\omega} \times (\lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3) \\ &= \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{L} = \vec{\Gamma}.\end{aligned}$$

Hence, component-wise (Euler's equations):

$$\begin{aligned}\lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_3)\, \omega_2 \omega_3 &= \Gamma_1, \\ \lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\, \omega_3 \omega_1 &= \Gamma_2, \\ \lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\, \omega_1 \omega_2 &= \Gamma_3.\end{aligned}$$

Application: In many rigid-body problems, $\Gamma_{1,2,3}$ are components in the rotating (body) frame, which can make them awkward to track. Euler's equations are especially useful for $\vec{\Gamma} = 0$, or for the case where one component is zero.

A spinning wing nut and spacecraft stability. In 1985, Vladimir Dzhanibekov observed a spinning wing nut（蝶形螺母）repeatedly flip in space.[3] Why can free rotation become unstable? The same stability question matters when selecting a spacecraft's spin axis for safe operation.[4]

The wing nut's flips begin with intermediate-axis instability.[3] For a rigid spacecraft in torque-free motion, spin about the smallest- or largest-inertia principal axis tolerates small disturbances, while intermediate-axis spin can develop into large tumbling and loss of antenna or instrument pointing. With internal energy dissipation, spin about the largest-inertia axis is preferred: it has the lowest rotational energy at fixed angular momentum.[4]

14.1.1 A symmetric top in a gravitational field

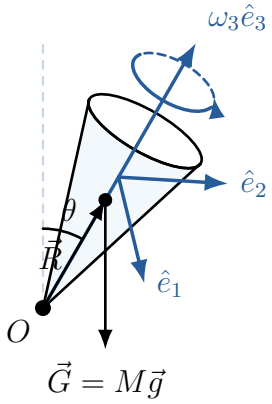


Fig. 2: A symmetric top: gravity acts at the center of mass, with lever arm $\vec{R}$ from the pivot.

Symmetric top spinning in a gravitational field. Let $\vec{R} = \ell\,\hat{e}_3$ be the vector from the pivot to the center of mass, and let $\vec{G} = M\vec{g}$ be the weight. The tilt θ is the angle between $\hat{e}_3$ and the upward vertical.

$$\vec{\Gamma} = \vec{R} \times \vec{G} = \ell\hat{e}_3 \times \vec{G} \Rightarrow \vec{\Gamma} \cdot \hat{e}_3 = 0.$$

For a symmetric top, $\lambda_1 = \lambda_2 \equiv \lambda$. Because gravity produces no torque about $\hat{e}_3$ through the contact point,

$$\lambda_3\dot{\omega}_3 = 0\,,$$

so ω_3 is constant.

Slow precession of a symmetric top

For a fast top, the most important fact is directional: torque changes angular momentum by

$$\frac{d\vec{L}}{dt} = \vec{\Gamma}.$$

If the spin angular momentum is large, gravity may change mainly the direction of $\vec{L}$ rather than its magnitude. This gives slow precession. For steady slow precession at tilt θ, with spin-dominated angular momentum $L \approx \lambda_3\omega_3$,

$$\Gamma = Mg\ell\sin\theta, \qquad \left|\frac{d\vec{L}}{dt}\right| = \Omega_{\mathrm{p}} L\sin\theta,$$

$$\Omega_{\mathrm{p}} \approx \frac{Mg\ell}{\lambda_3\omega_3}.$$

A large spin angular momentum turns slowly under the gravitational torque in this approximation. General motion can also include nutation（章动）, an oscillation of the tilt angle.[1]

What happens at the instant of release? Hold a spinning gyroscope with its axis horizontal, then release the axis. Does it initially fall? It does: the downward motion starts before steady precession is established. The axis then develops sideways motion and nutates. With dissipation, the nutation can decay and leave approximately steady precession with the axis slightly lower. The ideal rigid-body equations here contain no damping, so they also allow persistent nutation; the steady-precession formula describes a special motion.[1]

Video: MIT Physics Demo — Bicycle Wheel Gyroscope.

14.2 Torque-free Euler's equations

If $\vec{\Gamma} = 0$, then $\vec{L}$ is conserved in the lab (space) frame, but not necessarily in the body frame:

$$\lambda_1\dot{\omega}_1 = (\lambda_2 - \lambda_3)\,\omega_2\omega_3,$$

$$\lambda_2\dot{\omega}_2 = (\lambda_3 - \lambda_1)\,\omega_3\omega_1,$$

$$\lambda_3\dot{\omega}_3 = (\lambda_1 - \lambda_2)\,\omega_1\omega_2.$$

⋆ If we start with $\vec{\omega}$ along a principal axis, say $\hat{e}_1$:

$$\omega_2 = \omega_3 = 0 \text{ at } t = 0 \quad \Rightarrow \quad \begin{cases} \lambda_1\dot{\omega}_1 = 0, \\ \lambda_2\dot{\omega}_2 = 0, \\ \lambda_3\dot{\omega}_3 = 0, \end{cases}$$

so $\vec{\omega}$ is constant (pure spin about that axis).

⋆ If $\vec{\omega}$ is not along any principal axis and $\lambda_i \neq \lambda_j$ for $i \neq j$, the nonzero components generally evolve through the coupled Euler equations. A constant body-frame angular velocity for three distinct principal moments requires rotation about a principal axis. Thus although $\vec{L}$ is conserved, $\vec{\omega}$ is not.

14.2.1 Stability of rotation around a principal axis

Suppose at $t = 0$, $\vec{\omega} = \omega_3 \hat{e}_3$ with $\omega_1 = \omega_2 = 0$. Choose $\omega_3 > 0$ by orienting $\hat{e}_3$ along the initial spin. Assume three distinct positive principal moments. Now give a small perturbation in ω_1, ω_2. Will $\omega_{1,2}$ grow? From Euler's equations,

$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\,\omega_1 \omega_2 \sim \text{2nd-order small} \approx 0,$$

so $\omega_3 \approx$ const.

Linearize:

$$\begin{cases} \dot{\omega}_1 = \dfrac{\lambda_2 - \lambda_3}{\lambda_1}\,\omega_3\,\omega_2, \\ \dot{\omega}_2 = \dfrac{\lambda_3 - \lambda_1}{\lambda_2}\,\omega_3\,\omega_1. \end{cases}$$

Differentiating the first equation and substituting the second gives

$$\begin{aligned} \ddot{\omega}_1 &= \frac{\lambda_2 - \lambda_3}{\lambda_1}\,\omega_3 \dot{\omega}_2 \\ &= \frac{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2}\,\omega_3^2 \omega_1. \end{aligned}$$

This is a second-order equation with a constant coefficient. Try solutions

$$\omega_1(t) = A e^{\mu t}, \qquad \omega_2(t) = B e^{\mu t}.$$

Then

$$\begin{cases} A\mu = \dfrac{\lambda_2 - \lambda_3}{\lambda_1}\,\omega_3\,B, \\ B\mu = \dfrac{\lambda_3 - \lambda_1}{\lambda_2}\,\omega_3\,A, \end{cases}$$

Eliminating B,

$$\begin{aligned} \mu^2 A &= \frac{\lambda_2 - \lambda_3}{\lambda_1}\,\omega_3 (\mu B) \\ &= \frac{\lambda_2 - \lambda_3}{\lambda_1}\frac{\lambda_3 - \lambda_1}{\lambda_2}\,\omega_3^2 A. \end{aligned}$$

For a nonzero mode, therefore,

$$\mu^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2}\,\omega_3^2.$$

Case (1): If λ_3 lies between λ_1 and λ_2, then $\mu^2 > 0$. We get a pair of real solutions $\mu_{1,2} = \pm\mu_0$ with

$$\mu_0 = \omega_3 \sqrt{\frac{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2}} > 0.$$

Then

$$\omega_1(t) = A e^{\mu_0 t} + A' e^{-\mu_0 t},$$

$$\omega_2(t) = B e^{\mu_0 t} + B' e^{-\mu_0 t}.$$

The first-order equations leave only two independent constants. With

$$c = \frac{(\lambda_2 - \lambda_3)\omega_3}{\lambda_1}, \qquad B = \frac{\mu_0 A}{c}, \qquad B' = -\frac{\mu_0 A'}{c},$$

generic small perturbations contain a growing mode $\sim e^{\mu_0 t}$. The rotation is **unstable**. The specially chosen initial condition $A = 0$ lies entirely in the decaying mode; arbitrarily small departures can excite the growing mode.

Case (2): If λ_3 is either the largest or the smallest among $\lambda_1, \lambda_2, \lambda_3$, then $\mu = \pm i\mu_0$ with

$$\mu_0 = \frac{\omega_3}{\sqrt{\lambda_1 \lambda_2}} \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\,.$$

Write

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\mu_0 t},$$

which implies

$$A\, i\mu_0 = \frac{\lambda_2 - \lambda_3}{\lambda_1}\, \omega_3 B.$$

Solving for B and inserting μ_0,

$$\begin{aligned} B &= \frac{i\lambda_1 \mu_0}{(\lambda_2 - \lambda_3)\omega_3}\, A \\ &= \frac{i}{\sqrt{\lambda_1 \lambda_2}} \frac{\lambda_1}{\lambda_2 - \lambda_3} \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\, A. \end{aligned}$$

Hence

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = A \begin{pmatrix} 1 \\ \pm i \sqrt{\dfrac{\lambda_1(\lambda_1 - \lambda_3)}{\lambda_2(\lambda_2 - \lambda_3)}} \end{pmatrix} e^{i\mu_0 t},$$

where the sign depends on whether $\lambda_2 > \lambda_3$ (+) or $\lambda_2 < \lambda_3$ (-).

The complex conjugate gives the other solution with $e^{-i\mu_0 t}$. Writing $A = A' e^{i\varphi}$ and taking the real part,

$$\mathrm{Re}\big(A e^{i\mu_0 t}\big) = A' \cos(\mu_0 t + \varphi),$$

$$\mathrm{Re}\big(\pm i A e^{i\mu_0 t}\big) = \mp A' \sin(\mu_0 t + \varphi).$$

Hence the real solution is

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = A' \begin{pmatrix} \cos(\mu_0 t + \varphi) \\ \mp \sin(\mu_0 t + \varphi) \sqrt{\frac{\lambda_1}{\lambda_2} \frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3}} \end{pmatrix},$$

which is purely oscillatory ($\lambda_2 > \lambda_3 \rightarrow$"-"). This rotation is **stable**.

14.3 Torque-free symmetric top

For a torque-free symmetric top, let $\lambda_1 = \lambda_2 = \lambda$ and λ_3 possibly different. Choose the positive $\hat{e}_3$ direction so $\omega_3 > 0$. Then $\omega_3 = \text{const}$.

A standard solution is

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \omega_0 \cos\left(\dfrac{\omega_3(\lambda-\lambda_3)}{\lambda}t\right) \\ -\omega_0 \sin\left(\dfrac{\omega_3(\lambda-\lambda_3)}{\lambda}t\right) \\ \omega_3 \end{pmatrix}.$$

For an "egg-shaped" (prolate, 长旋转椭球形) configuration with $\lambda_3 < \lambda = \lambda_1 = \lambda_2$, we can write

$$\vec{\omega} = \omega_0 \cos(\Omega_b t)\,\hat{e}_1 - \omega_0 \sin(\Omega_b t)\,\hat{e}_2 + \omega_3 \hat{e}_3, \quad \Omega_b = \frac{\omega_3}{\lambda}(\lambda - \lambda_3).$$

In the body frame, $\vec{\omega}$ precesses around $-\hat{e}_3$.

The angular momentum is

$$\vec{L} = \lambda\,\omega_0 \cos(\Omega_b t)\,\hat{e}_1 - \lambda\,\omega_0 \sin(\Omega_b t)\,\hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3.$$

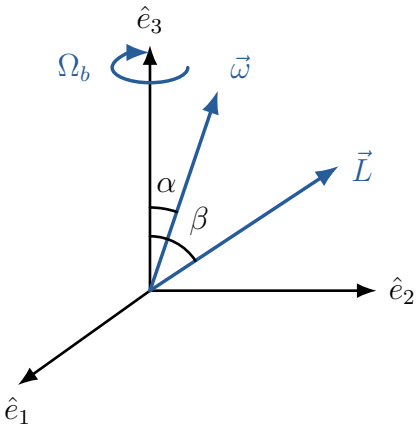


Fig. 3: For a prolate symmetric top, angular momentum has a larger tilt than angular velocity.

Define the tilt angles:

$$\tan\alpha = \frac{\omega_0}{\omega_3}, \qquad \tan\beta = \frac{\omega_0}{\omega_3}\frac{\lambda}{\lambda_3} \quad \Rightarrow \quad \beta > \alpha.$$

In the body frame, $\vec{\omega}$ and $\vec{L}$ both precess around $-\hat{e}_3$ with angular speed Ω_b. In space-fixed directions, $\vec{L}$ remains constant and the symmetry axis $\hat{e}_3$ moves around it.

One motion, two views

The relation between the two precession rates follows directly from

$$\vec{L} = \lambda\vec{\omega} + (\lambda_3 - \lambda)\omega_3\hat{e}_3.$$

Taking the cross product with $\hat{e}_3$ gives

$$\left(\frac{d\hat{e}_3}{dt}\right)_{\text{space}} = \vec{\omega} \times \hat{e}_3 = \frac{\vec{L}}{\lambda} \times \hat{e}_3.$$

Thus the symmetry axis precesses about the fixed $\vec{L}$ with

$$\vec{\Omega}_s = \frac{\vec{L}}{\lambda}, \qquad \Omega_s = \frac{L}{\lambda}.$$

This space-frame rate and the body-frame rate $\Omega_b = (\lambda - \lambda_3)\omega_3/\lambda$ describe different moving directions.

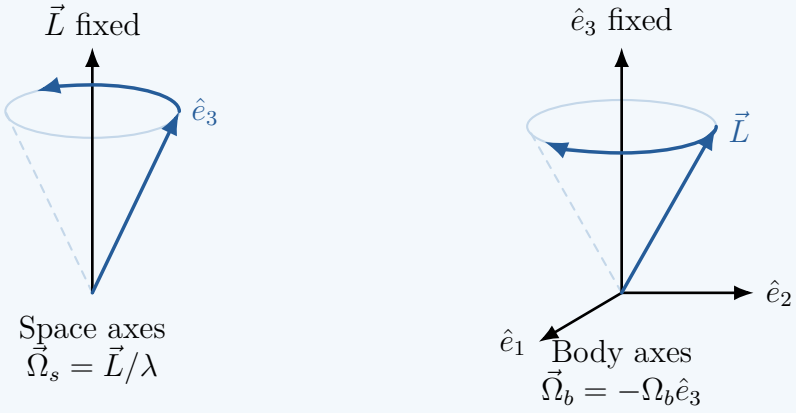


Fig. 4: Space view: $\vec{L}$ is fixed and the symmetry axis moves around it. Body view: the axes are fixed and the components of $\vec{L}$ rotate around $-\hat{e}_3$ for a prolate top.

Energy and angular momentum: a geometric view

For torque-free rotation, write $L_i = \lambda_i \omega_i$ in the body principal axes. Conservation gives two surfaces in the space of components (L_1, L_2, L_3):

$$L_1^2 + L_2^2 + L_3^2 = L^2 \quad \text{(sphere)},$$

$$\frac{L_1^2}{2\lambda_1} + \frac{L_2^2}{2\lambda_2} + \frac{L_3^2}{2\lambda_3} = E \quad \text{(ellipsoid)}.$$

The tip of the body-component vector follows their intersection. Euler's equations determine its speed along the curve.[5]

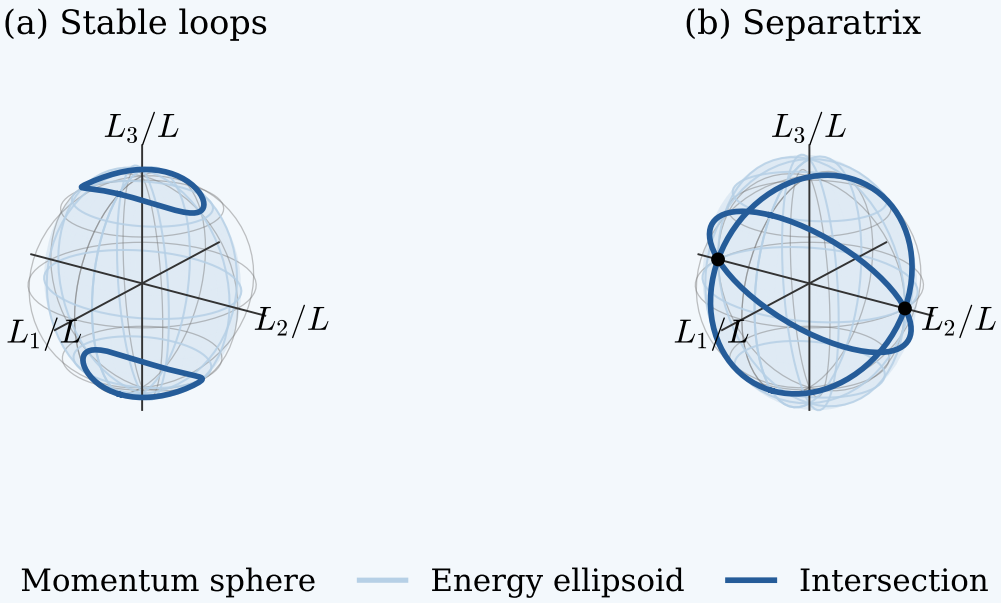


Fig. 5: Intersections for $\lambda_1 : \lambda_2 : \lambda_3 = 2 : 3 : 4$. Left: closed loops near the largest-inertia axis. Right: the separatrix at $E = L^2/(2\lambda_2)$; black points mark rotations about $\pm\hat{e}_2$.

For $\lambda_1 < \lambda_2 < \lambda_3$, the minimum and maximum energies at fixed L occur on the largest- and smallest-inertia axes. Nearby intersections are small closed loops: perturbations remain small and oscillate. At the intermediate axis, the separatrix (分界轨道) branches meet at unstable steady rotations, which a moving trajectory approaches only asymptotically. Nearby trajectories can leave the axis by a large angle, producing flips.
For a symmetric top, L_3 is constant and the intersection is a circle at fixed L_3. This is the body-frame precession discussed above. In space, $\vec{L}$ stays fixed while the body axes move around it.

References

[1] R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. I, Sec. 20–3, *The gyroscope*. https://www.feynmanlectures.caltech.edu/I_20.html

[2] Original 2025 Lecture 14, *Rigid body (II)*. The Euler-equation and rigid-body rotation sections retain the source body axes, lever arm, and tilt angles. The final dual-view diagram accompanies the derived relation $\Omega_s = L/\lambda$.

[3] P. M. Trivailo and H. Kojima, *Enhancement of the attitude dynamics capabilities of the spinning spacecraft using inertial morphing*, The Aeronautical Journal **124**, 838–871 (2020), Sec. 6. https://doi.org/10.1017/aer.2019.145

[4] W. Hewitt Phillips, *Journey Into Space Research*, NASA/SP-2005-4540 (2005), Chapter 3, pp. 17–21. https://www.nasa.gov/wp-content/uploads/2023/04/sp-4540.pdf

[5] G. J. Sussman and J. Wisdom, *Structure and Interpretation of Classical Mechanics*, 2nd ed., Sec. 2.8.2, *Qualitative features of free rigid body motion*. MIT Press: Chapter 2, Rigid bodies.