Lecture 13: Inertia Tensor

13.1 Inertia tensor for rotation about a point

Take the origin at a point fixed in an inertial frame, or at the center of mass with velocities measured relative to it. The relation $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ then describes the rotational motion.

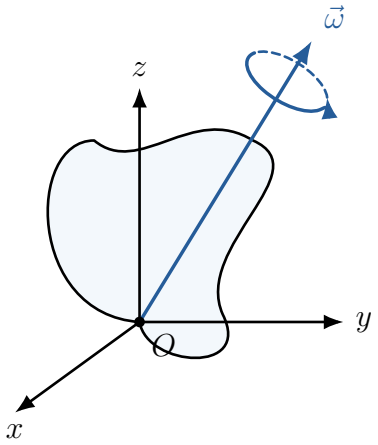


Fig. 1: Angular velocity about an arbitrary axis through the chosen origin.

For rotation about the z-axis,

$$L_x = I_{xz}\omega, \qquad L_y = I_{yz}\omega, \qquad L_z = I_{zz}\omega = I_z\omega.$$

For an arbitrary $\vec{\omega}$,

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha),$$

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = r^2\vec{\omega} - \vec{r}(\vec{\omega} \cdot \vec{r}),$$

Thus

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \mathbf{I}\vec{\omega},$$

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \sum_\alpha m_\alpha \begin{pmatrix} y_\alpha^2 + z_\alpha^2 & -x_\alpha y_\alpha & -x_\alpha z_\alpha \\ -y_\alpha x_\alpha & z_\alpha^2 + x_\alpha^2 & -y_\alpha z_\alpha \\ -z_\alpha x_\alpha & -z_\alpha y_\alpha & x_\alpha^2 + y_\alpha^2 \end{pmatrix}.$$

Here $I_{ab} = I_{ba}$. In I_{ab}, a labels the component of $\vec{L}$ and b the component of $\vec{\omega}$; for example, $I_{xz}\omega_z$ contributes to L_x. Off-diagonal entries allow $\vec{L}$ and $\vec{\omega}$ to point in different directions.

Equivalently, $L_a = \sum_b I_{ab}\omega_b$, with $a, b \in \{x, y, z\}$ and

$$I_{ab} = \sum_\alpha m_\alpha \left(r_\alpha^2 \delta_{ab} - x_{\alpha a} x_{\alpha b}\right) = \int \rho \left(r^2 \delta_{ab} - x_a x_b\right) dV.^1$$

[1] $\delta_{ab} = 1$ for $a = b$ and $\delta_{ab} = 0$ for $a \neq b$.

13.2 Kinetic energy of fixed-point rotation

For rigid rotation,

$$\vec{v} = \vec{\omega} \times \vec{r} \quad \Rightarrow \quad v^2 = \omega^2 r^2 - (\vec{\omega} \cdot \vec{r})^2 \,.$$

Thus

$$\begin{aligned}
E_k &= \frac{1}{2} \sum_\alpha m_\alpha v_\alpha^2 \\
&= \frac{1}{2} \sum_\alpha m_\alpha \Big[\omega^2 r_\alpha^2 - (\vec{\omega} \cdot \vec{r}_\alpha)^2 \Big] \\
&= \frac{1}{2} \sum_\alpha m_\alpha \Big[(\omega_x^2 + \omega_y^2 + \omega_z^2)(x_\alpha^2 + y_\alpha^2 + z_\alpha^2) \\
&\quad - (\omega_x x_\alpha + \omega_y y_\alpha + \omega_z z_\alpha)^2 \Big] \\
&= \frac{1}{2} \sum_\alpha m_\alpha \Big[\cancel{\omega_x^2 x_\alpha^2} + \cancel{\omega_y^2 y_\alpha^2} + \cancel{\omega_z^2 z_\alpha^2} \\
&\quad + \omega_x^2 (y_\alpha^2 + z_\alpha^2) + \omega_y^2 (z_\alpha^2 + x_\alpha^2) + \omega_z^2 (x_\alpha^2 + y_\alpha^2) \\
&\quad - \cancel{\omega_x^2 x_\alpha^2} - \cancel{\omega_y^2 y_\alpha^2} - \cancel{\omega_z^2 z_\alpha^2} \\
&\quad - 2\omega_x \omega_y x_\alpha y_\alpha - 2\omega_y \omega_z y_\alpha z_\alpha - 2\omega_z \omega_x z_\alpha x_\alpha \Big] \\
&= \frac{1}{2} \Big[I_{xx} \omega_x^2 + I_{yy} \omega_y^2 + I_{zz} \omega_z^2 \\
&\quad + 2I_{xy} \omega_x \omega_y + 2I_{yz} \omega_y \omega_z + 2I_{zx} \omega_z \omega_x \Big].
\end{aligned}$$

In matrix notation,

$$\begin{aligned}
E_k &= \frac{1}{2} \boldsymbol{\omega}^\mathsf{T} \mathbf{I} \boldsymbol{\omega} = \frac{1}{2} \sum_{a=x,y,z} \sum_{b=x,y,z} \omega_a I_{ab} \omega_b \\
&= \frac{1}{2} \omega_a I_{ab} \omega_b = \frac{1}{2} \vec{\omega} \cdot (\mathbf{I} \vec{\omega}) = \frac{1}{2} \vec{\omega} \cdot \vec{L}.
\end{aligned}$$

Here $a, b \in \{x, y, z\}$, and repeated indices imply summation.

13.3 Principal axes

The real symmetric matrix $\mathbf{I}$ has orthonormal eigenvectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$:

$$\mathbf{I}\hat{e}_i = \lambda_i \hat{e}_i, \qquad Q = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix}, \qquad Q^{\mathsf{T}} Q = \mathbf{1}.$$

For rotation about one of these axes,

$$\vec{\omega} = \omega_i \hat{e}_i \quad \Rightarrow \quad \vec{L} = \lambda_i \omega_i \hat{e}_i \parallel \vec{\omega}.$$

In the principal-axis coordinates, both the inertia matrix and the angular-velocity components transform:

$$\mathbf{I}_{\text{pr}} = Q^{\mathsf{T}} \mathbf{I} Q = \text{diag}(\lambda_1, \lambda_2, \lambda_3),$$

$$\boldsymbol{\omega}_{\text{pr}} = Q^{\mathsf{T}} \boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \qquad \boldsymbol{\omega} = Q \boldsymbol{\omega}_{\text{pr}}.$$

Therefore,

$$\begin{aligned} E_k &= \frac{1}{2} (Q\boldsymbol{\omega}_{\text{pr}})^{\mathsf{T}} \mathbf{I} (Q\boldsymbol{\omega}_{\text{pr}}) \\ &= \frac{1}{2} \boldsymbol{\omega}_{\text{pr}}^{\mathsf{T}} (Q^{\mathsf{T}} \mathbf{I} Q) \boldsymbol{\omega}_{\text{pr}} \\ &= \frac{1}{2} \left(\lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2 \right), \end{aligned}$$

where $(\omega_1, \omega_2, \omega_3)$ are the components of $\vec{\omega}$ along the principal axes.

Example: a uniform cuboid with the origin at its center of mass and the x, y, z axes parallel to its edges:

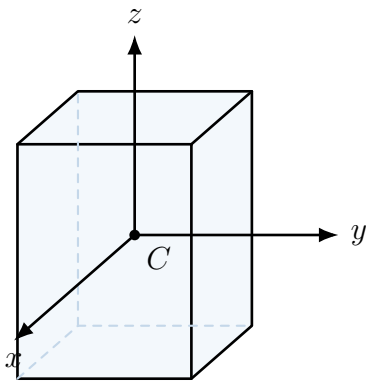


Fig. 2: A cuboid with principal axes through its center of mass.

$$I_{xy} = I_{yz} = I_{zx} = 0, \qquad I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}.$$

The x, y, z axes are principal axes.

13.4 Example: inertia tensor of a solid cube

(1) Around the corner (take the cube of side a with one corner at the origin, uniform density ρ)

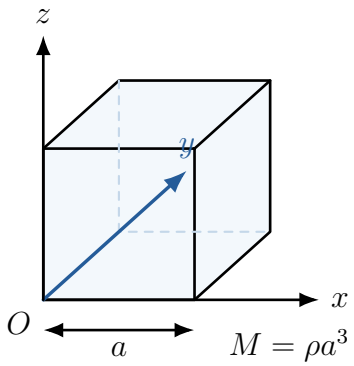


Fig. 3: Uniform cube of side a with the origin at one corner.

$$
\begin{aligned}
I_{xx} &= \rho \int_0^a dx \int_0^a dy \int_0^a dz\,(y^2+z^2) \\
&= \rho a \left(\frac{a^3}{3} a + a \frac{a^3}{3} \right) \\
&= \frac{2}{3}\rho a^5 = \frac{2}{3} M a^2,
\end{aligned}
$$

where $M = \rho a^3$ is the total mass. By symmetry,

$$
I_{yy} = I_{zz} = I_{xx}\,.
$$

For the products of inertia (off-diagonal terms),

$$
\begin{aligned}
I_{xy} &= -\int_0^a dx \int_0^a dy \int_0^a dz\,\rho\, xy = -\rho \left(\int_0^a dx\, x \right) \left(\int_0^a dy\, y \right) \left(\int_0^a dz \right) \\
&= -\rho \left(\frac{a^2}{2} \right) \left(\frac{a^2}{2} \right) (a) = -\rho\, \frac{a^5}{4} = -\frac{a^2}{4} M\,.
\end{aligned}
$$

Similarly $I_{yz} = I_{zx} = I_{xy}$ by symmetry for this corner choice. Therefore,

$$
I = M a^2 \begin{bmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{bmatrix}.
$$

If the cube rotates around the z-axis,

$$\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \quad \Rightarrow \quad \vec{L} = Ma^2\omega \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{2}{3} \end{bmatrix}.$$

The nonzero transverse components make $\vec{L}$ and $\vec{\omega}$ nonparallel.

(2) If the cube rotates around its center:

Now the origin is at the center of the cube, so $x, y, z \in [-a/2, a/2]$.

$$\begin{aligned} I_{xx} &= \rho \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz\, (y^2 + z^2) \\ &= \rho a \left(\frac{a^3}{12} a + a \frac{a^3}{12} \right) \\ &= \frac{\rho a^5}{6} = \frac{Ma^2}{6}. \end{aligned}$$

so

$$I_{xx} = I_{yy} = I_{zz} = \frac{Ma^2}{6}, \qquad I_{xy} = I_{yz} = I_{zx} = 0,$$

and therefore

$$I = \frac{Ma^2}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \vec{L} = \frac{Ma^2}{6} \vec{\omega}.$$

The inertia tensor I depends on the choice of origin and the orientation of the axes.

The cube's hidden isotropy

Would a central axis through a vertex give a different moment of inertia from one through a face? For a unit direction $\hat{n}$,

$$I_{\hat{n}} = \hat{n} \cdot \mathbf{I}\hat{n} = \frac{Ma^2}{6}\, \hat{n} \cdot \hat{n} = \frac{Ma^2}{6}.$$

Every axis through the center gives the same result. The cube has only discrete shape symmetries, yet its second mass moment is isotropic: $\mathbf{I} = (Ma^2/6)\mathbf{1}$. The rotational kinetic energy of a rigid body depends only on the second moments of its mass distribution.

References

[1] Original 2025 Lecture 14, *Rigid body (II)*. Inertia-tensor and rotational-energy sections; the source axes and cube/cuboid geometry are retained.