Lecture 12: Rigid body (I) — fixed-axis rotation

- Rotation around a fixed axis
- Moment of inertia
- Equation of motion of fixed axis rotation

12.1 What is a rigid body?

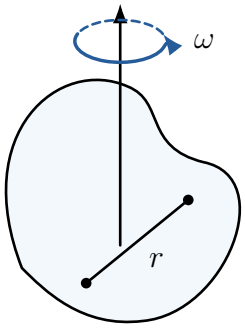


Fig. 1: A rigid body preserves all interparticle distances during rotation.

Rigid-body idealization.
The distance between any two points is fixed during the motion, i.e. no shape deformation.

How many degrees of freedom are needed to describe rigid-body motion? Let us count the independent coordinates.

Particles	Independent coordinates	Rigid configuration
1	3	Point
2	$6-1=5$	Line segment
3	$9-3=6$	Noncollinear triangle
4	6	Nondegenerate tetrahedron
$N \geq 3$	6	General noncollinear body

3 translational degrees of freedom
3 rotational degrees of freedom

12.2 Kinematics of fixed-axis rotation

Measure $\vec{r}$ from a fixed point O on the rotation axis. For the instantaneous velocity at a point, write $\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$. $\vec{r}_{\parallel}$ is the component parallel to the rotation axis, $\vec{r}_{\perp}$ is the component perpendicular to the axis.

$$\vec{v} = \vec{\omega} \times \vec{r}_{\perp} = \vec{\omega} \times \left(\vec{r}_{\perp} + \vec{r}_{\parallel}\right) = \vec{\omega} \times \vec{r}$$

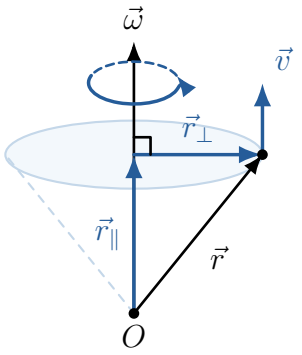


Fig. 2: Only the displacement perpendicular to the fixed axis contributes to $\vec{v} = \vec{\omega} \times \vec{r}$.

12.3 Angular momentum about the fixed axis

$$\vec{L} = \sum_{i=1}^{N} \vec{l}_i = \sum_{i=1}^{N} m_i \vec{r}_i \times \vec{v}_i$$

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

$$\Rightarrow \quad \vec{L} = \sum_{i=1}^{N} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

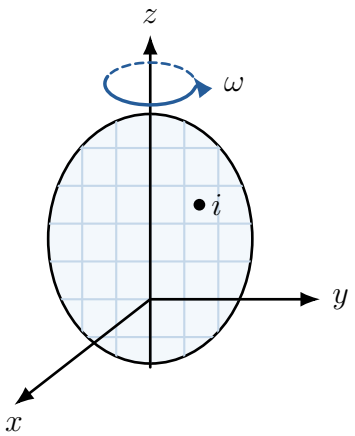


Fig. 3: A mass element of a body rotating about the fixed z axis.

In our case[1]

$$\vec{\omega} = \omega\,\hat{z}, \qquad \vec{r}_i = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}$$

$$\begin{aligned}\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) &= \vec{\omega}\, r_i^2 - \vec{r}_i\ (\vec{\omega} \cdot \vec{r}_i) \\ &= \left(x_i^2 + y_i^2 + z_i^2\right) \omega\,\hat{z} - \omega\left(z_i x_i \hat{x} + z_i y_i \hat{y} + z_i^2 \hat{z}\right) \\ &= -\omega z_i x_i \hat{x} - \omega z_i y_i \hat{y} + \omega\left(x_i^2 + y_i^2\right)\hat{z}\end{aligned}$$

$$L_z = \sum_{i=1}^{N} m_i\,\omega\,\left(x_i^2 + y_i^2\right) = I_z\,\omega, \quad \text{where } I_z = \sum_i m_i\left(x_i^2 + y_i^2\right)$$

For a continuous mass distribution, the moment of inertia is

$$I_z = \int \rho(x,y,z)\left(x^2 + y^2\right)\,dx\,dy\,dz,$$

where ρ is the volume mass density.

$$\left.\begin{aligned} L_x &= -\left(\textstyle\sum_i m_i x_i z_i\right)\omega = I_{xz}\,\omega \neq 0 \\ L_y &= -\left(\textstyle\sum_i m_i y_i z_i\right)\omega = I_{yz}\,\omega \neq 0 \end{aligned}\right\} \quad \text{in general}$$

Here $I_{xz} \equiv -\sum_i m_i x_i z_i$ and $I_{yz} \equiv -\sum_i m_i y_i z_i$.

Example: If the fixed axis is not a principal axis of inertia（惯量主轴）, $\vec{L}$ generally has transverse components and is not parallel to $\vec{\omega}$.

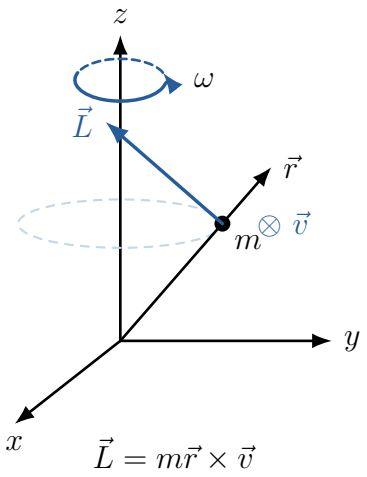


Fig. 4: For rotation about a nonprincipal axis, $\vec{L}$ can have components perpendicular to $\vec{\omega}$.

12.4 Kinetic energy

$$E_k = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i\,\omega^2\left(x_i^2 + y_i^2\right) = \frac{I_z}{2}\,\omega^2$$

$$\vec{v}_i = -\omega y_i \hat{x} + \omega x_i \hat{y}$$

[1] Vector triple-product identity: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$.

12.5 Parallel axis theorem

If the rotation axis does not pass the center of mass, denote the distance between the mass center and the rotation axis as r_c, then

$$I_z = I_{cz} + Mr_c^2,$$

where I_{cz} is the moment of inertia about the parallel axis through the center of mass. In the following proof, $\vec{r}_i$, $\vec{r}_c$, and $\vec{r}_i'$ denote projections onto the plane perpendicular to these axes; their lengths are distances from the axes, not three-dimensional distances from an arbitrary origin.

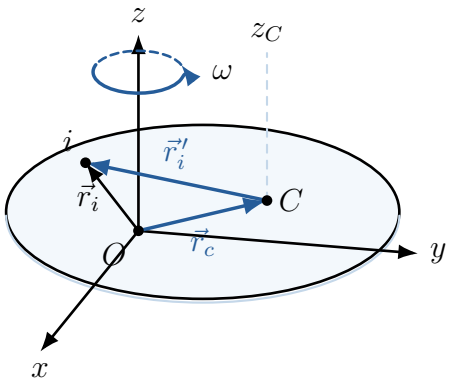


Fig. 5: The parallel-axis theorem uses transverse displacements from two parallel axes.

Write

$$\vec{r}_i = \vec{r}_c + \vec{r}_i', \qquad \vec{r}_i' \text{ is the displacement of "}i\text{th point" relative to } C$$

$$I_z = \sum_i m_i \left|\vec{r}_c + \vec{r}_i'\right|^2 = \sum_i m_i \left(r_c^2 + r_i'^2 + 2\vec{r}_c \cdot \vec{r}_i'\right)$$

$$= Mr_c^2 + I_{cz} + 2\vec{r}_c \cdot \sum_i m_i \vec{r}_i'$$

By definition

$$\sum_i m_i \vec{r}_i = \left(\sum_i m_i\right) \vec{r}_c \quad \Rightarrow \quad \sum_i m_i \vec{r}_i' = 0$$

Hence

$$I_z = I_{cz} + Mr_c^2.$$

Then

$$E_k = \frac{1}{2} I_z \omega^2 = \frac{1}{2} I_{cz} \omega^2 + \frac{1}{2} M \left(r_c \omega\right)^2.$$

For rotation about an axis away from the center of mass, the kinetic energy is the sum of rotation about the center of mass and translation of the center of mass. Circular translation alone preserves the orientation of the body; rotation also changes that orientation.



Circular translation
Orientation fixed

Rotation about an axis away
from the center of mass

Fig. 6: Motion of the center of mass and rotation about the center of mass are distinct contributions.

12.5.1 Examples of moment of inertia

(1) Uniform thin rod — transverse axis through the center of mass. Here $\rho = M/L$ is linear mass density.

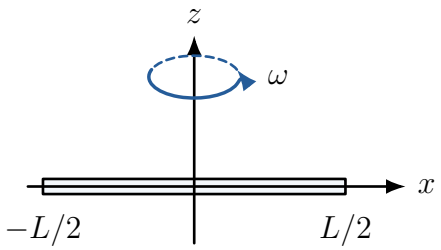


Fig. 7: A thin rod rotates about a transverse axis through its midpoint.

$$I_c = \int_{-L/2}^{L/2} \rho x^2 \, dx = \frac{\rho}{3} x^3 \Big|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{\rho}{12} L^3 = \frac{(\rho L) L^2}{12} = \frac{M}{12} L^2$$

If the rotation axis is located at one end, then

$$I_z = I_{cz} + M \left(\frac{L}{2} \right)^2 = \frac{M}{3} L^2$$

(2) Uniform circular disk — normal axis through its center. Here $\rho = M/(\pi R^2)$ is surface mass density.

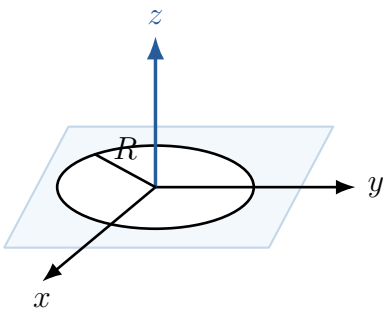


Fig. 8: A disk rotates about the normal through its center.

$$I_z = \int dx \int dy\ \rho\left(x^2+y^2\right) = \rho \int_0^R r\,dr \int_0^{2\pi} d\theta\ r^2$$
$$= 2\pi\rho \cdot \frac{R^4}{4} = \left(\pi R^2 \rho\right)\frac{R^2}{2} = \frac{1}{2}MR^2$$

(3) Uniform rectangular plate — normal axis through its center. Here $\rho = M/(ab)$ is surface mass density.

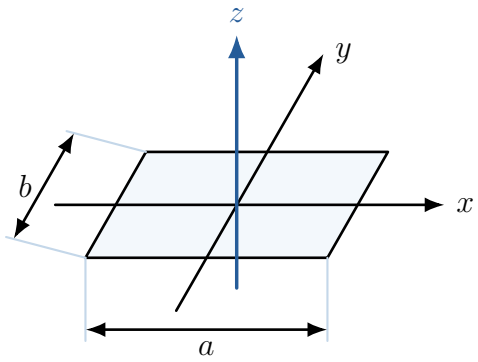


Fig. 9: A rectangular plate rotates about the normal through its center.

$$I_z = \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\ \rho\left(x^2+y^2\right)$$
$$= \rho\left(b\cdot\frac{x^3}{3}\bigg|_{-\frac{a}{2}}^{\frac{a}{2}} + a\cdot\frac{y^3}{3}\bigg|_{-\frac{b}{2}}^{\frac{b}{2}}\right)$$
$$= \rho\left(\frac{ba^3}{12}+\frac{ab^3}{12}\right) = \frac{\rho ab}{12}\left(a^2+b^2\right) = \frac{M}{12}\left(a^2+b^2\right)$$

(4) Uniform solid sphere — rotation about a diameter. Here $\rho = 3M/(4\pi R^3)$ is volume mass density.

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$

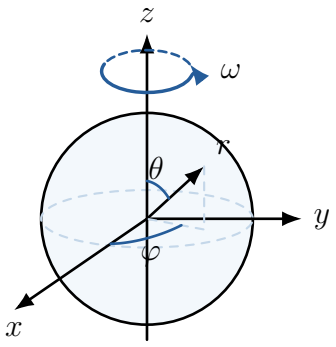


Fig. 10: A uniform solid sphere rotates about a diameter; the perpendicular distance is $r\sin\theta$.

$$I = \int \rho\left(r^2\sin^2\theta\right) r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \rho\int_0^R r^4\, dr\int_0^\pi \sin^3\theta\, d\theta\int_0^{2\pi} d\varphi$$

$$\int_0^\pi \sin^3\theta\, d\theta = -\int_0^\pi \sin^2\theta\, d(\cos\theta) = \int_{-1}^1 (1-x^2)dx$$

$$= \left[x - \frac{x^3}{3}\right]_{-1}^1 = \frac{4}{3}$$

$$I = \rho\cdot\frac{R^5}{5}\cdot\frac{4}{3}\cdot 2\pi$$

$$= \left(\frac{4}{3}\pi R^3\rho\right)\frac{2}{5}R^2 = \frac{2}{5}MR^2.$$

12.6 Equation of motion for fixed-axis rotation

Take a rotation axis fixed in an inertial frame, with $\vec{\omega} = \omega\hat{z}$.

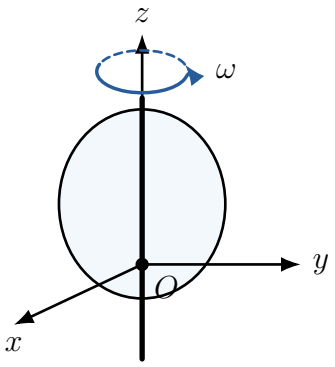


Fig. 11: Any fixed point on the rotation axis can be used to calculate the axial angular momentum and torque.

Let's prove

$$\frac{dL_z}{dt} = \tau_z \quad \leftarrow \text{external torque}$$

We can choose any fixed point O on the rotation axis as the origin. We proved before

$$\frac{d}{dt}\vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{\text{ex}}$$

We take the z-component:

(1) $L_z = I_z\omega$ and

$$I_z = \int dx\, dy\, dz\ \rho\left(x^2 + y^2\right)$$

i.e. L_z does not depend on the location of "O" on the axis.

(2)

$$\left(\vec{r}_i \times \vec{F}_i^{\text{ex}}\right)_z = x_i F_i^y - y_i F_i^x$$

which does not depend on the z-location of "O" either. Hence, we have

$$\boxed{\frac{dL_z}{dt} = \tau_z}$$

12.6.1 Example: rolling without slipping

For rolling, the center of mass moves. We use $\sum \vec{F} = M\vec{a}_C$ for its translation and $\tau_C = I\, d\omega/dt$ for rotation about the center of mass.

Let I be the moment of inertia about the center of mass and define the dimensionless shape factor（形状因子）

$$\kappa = \frac{I}{MR^2}.$$

(1) Constraint: Take positive motion down the incline; choose ω so $a = R\, d\omega/dt$. Static friction points up the incline when it supplies the torque for rolling.

$$\vec{v}_p = \vec{v}_0 + \vec{\omega} \times \vec{r}_{op} = 0$$

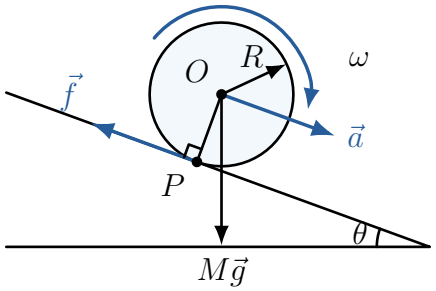


Fig. 12: Rolling down an incline: static friction provides the torque about the center of mass.

$$\Rightarrow v_0 = \omega R \Rightarrow a = R\frac{d\omega}{dt}$$

About the center of mass:

$$f \cdot R = I\frac{d\omega}{dt}$$
$$Mg\sin\theta - f = Ma$$

$$\Rightarrow Mg\sin\theta - I\frac{d\omega}{dt}\cdot\frac{1}{R} = Ma$$

$$Mg\sin\theta = \left(\frac{I}{R^2} + M\right)a$$
$$\Rightarrow a = \frac{1}{1+\kappa}\, g\sin\theta$$

(2) Energy conservation. Let h be the vertical distance descended from rest; static friction does no work on a body rolling without slipping on a stationary rigid plane.

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2$$

$$\Rightarrow g\frac{dh}{dt} = v\frac{dv}{dt}(1+\kappa)$$

$$\frac{dh}{dt} = v\sin\theta, \quad a = \frac{dv}{dt} \Rightarrow a = \frac{g\sin\theta}{1+\kappa}$$

Conditions for rolling: (1) $|f| \leq \mu_s M g \cos\theta$, hence, μ_s cannot be too small to achieve rolling without slipping.
(2) The largest angle for rolling without slipping

$$g\sin\theta\left(1-\frac{1}{1+\kappa}\right) \leqslant g\cos\theta\cdot\mu_s$$
$$\Rightarrow \tan\theta \leqslant \frac{\mu_s\,(1+\kappa)}{\kappa} = \mu_s\left(\frac{MR^2}{I}+1\right)$$

A race down the same slope

A thin ring, a uniform disk, and a uniform solid ball start from rest at the same height on a straight incline. All roll without slipping. Which reaches the bottom first? With the same radius R, their shape factors give

Body	κ	$a/(g\sin\theta)$	$v_{\text{bottom}}^2/(2gh)$
Ring	1	1/2	1/2
Disk	1/2	2/3	2/3
Solid ball	2/5	5/7	5/7

The ball arrives first, followed by the disk and then the ring. A larger κ puts a greater share of the released gravitational energy into rotation. The mass cancels, and for these shapes the radius cancels too. This comparison requires sufficient static friction for each body and neglects rolling resistance.

12.6.2 Example: a cylinder on a rug

Starting with both the rug（地毯）and the cylinder at rest, pull the horizontal rug with acceleration $\vec{a} = -a\hat{x}$, where $a > 0$, while the cylinder rolls on it without slipping. Let I be its central moment of inertia and R its radius.

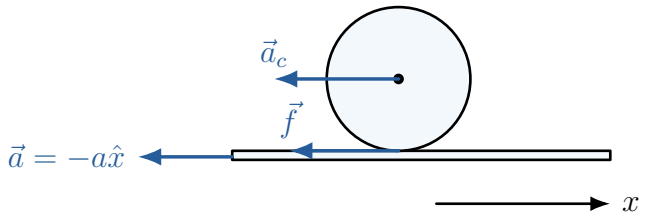


Fig. 13: A rug accelerates leftward; friction accelerates the cylinder leftward in the laboratory frame.

Then describe the motion of the cylinder.
We use the non-inertial frame of the rug.

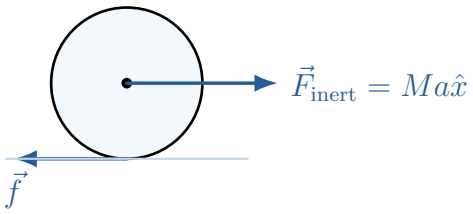


Fig. 14: In the rug frame, the inertial force points right and static friction points left.

In this frame, the inertial force is

$$\vec{F}_{\text{inert}} = Ma\hat{x}$$

Let $f \geq 0$ denote the friction magnitude, so $\vec{f} = -f\hat{x}$. Take clockwise rotation as positive and a' positive to the right in the rug frame. The torque about the cylinder center gives

$$\frac{d\omega}{dt} = \frac{a'}{R} = \frac{f \cdot R}{I}$$

and

$$\vec{F}_{\text{inert}} + \vec{f} = M\vec{a}'$$

a' is the acceleration of the cylinder center of mass in the rug frame.

$$\Rightarrow \quad Ma' = Ma - M\kappa a'$$

$$\Rightarrow \quad a' = \frac{a}{1+\kappa} = \frac{2}{3}a \quad \text{if } \kappa = \frac{1}{2}$$

The rug acceleration $\vec{a} = -a\hat{x} \Rightarrow$ the cylinder

$$\vec{a}_c = \vec{a} + \vec{a}' = -\frac{\kappa}{1+\kappa}a\hat{x}$$

$$\Rightarrow \vec{a}_c = -\frac{a}{3}\hat{x}, \quad \text{and } \vec{f} = Ma'\hat{x} - Ma\hat{x} = -\frac{M}{3}a\hat{x}$$

References

[1] Original 2025 Lecture 13, *Rigid body (I)*. Diagrams redrawn with the original axes, mass geometry, rolling contact, and force directions. Figure 6 reconstructs the circular-translation and rotation comparison from the diagram and surrounding text; the source PDF clips part of its handwritten explanation.