Lecture 11: Angular momentum conservation

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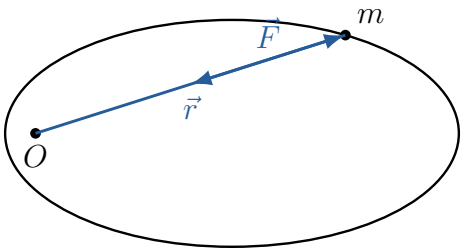
11.1 Recap: conservation laws from symmetry (elementary)

- Energy conservation ↔ time-translation symmetry.
- Linear momentum conservation ↔ space-translation symmetry.
  Every point in space is equivalent; an isolated body retains its velocity.
- Angular momentum conservation ↔ spatial isotropy (空间各向同性, rotational symmetry).

11.2 A particle in a central force field

Choose a fixed origin in an inertial frame. A central force about this origin has the form

$$\vec{F}(\vec{r}) = f(r)\hat{e}_r.$$



$$\vec{r} \times \vec{F} = 0$$

Fig. 1: A central force points along the line joining the particle and the fixed force center. Its moment about that center is zero.

Define angular momentum and torque (力矩) by

$$\vec{L} = \vec{r} \times \vec{p}, \qquad \vec{\tau} = \vec{r} \times \vec{F}.$$

For a particle of constant mass,

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} \\ &= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F} \\ &= \vec{\tau}. \end{aligned}$$

For a central force, $\vec{\tau} = 0$ and $\vec{L}$ is constant. The motion lies in a plane perpendicular to $\vec{L}$ when $L \neq 0$. Moreover,

$$\frac{dA}{dt} = \frac{L}{2m},$$

which is the constant areal velocity in Kepler's second law.[1] If $L = 0$, the motion is radial.

11.3 Angular momentum of several particles

11.3.1 Orbital and internal angular momentum

For particles $\alpha = 1, \ldots, N$ of constant masses, define

$$M = \sum_\alpha m_\alpha, \qquad \vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha.$$

$$\vec{r}'_\alpha = \vec{r}_\alpha - \vec{R}.$$

The axes attached to the center of mass are taken parallel to the inertial axes, without rotation. Then $\dot{\vec{r}}_\alpha = \dot{\vec{R}} + \dot{\vec{r}}'_\alpha$.

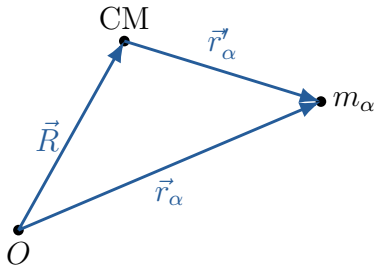


Fig. 2: Position-vector decomposition: $\vec{r}_\alpha = \vec{R} + \vec{r}'_\alpha$.

The center-of-mass definitions imply

$$\sum_\alpha m_\alpha \vec{r}'_\alpha = 0, \qquad \sum_\alpha m_\alpha \dot{\vec{r}}'_\alpha = 0.$$

With $\vec{P}_\alpha = m_\alpha \dot{\vec{r}}_\alpha$ and $\vec{P}_{\text{tot}} = M\dot{\vec{R}}$,

$$\vec{\ell}_\alpha = \vec{r}_\alpha \times \vec{P}_\alpha,$$

$$\vec{L}_{\text{tot}} = \sum_\alpha \vec{\ell}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{P}_\alpha.$$

[1] In an infinitesimal interval dt, $d\vec{r} = \vec{v}\,dt$, and the swept triangle has area $dA = \frac{1}{2}|\vec{r} \times d\vec{r}|$. Hence $dA/dt = \frac{1}{2}|\vec{r} \times \vec{v}| = L/(2m)$, using $L = m|\vec{r} \times \vec{v}|$. Here L denotes the angular-momentum magnitude.

$$
\begin{aligned}
\vec{L}_{\text{tot}} &= \sum_{\alpha} (\vec{R} + \vec{r}'_{\alpha}) \times \vec{P}_{\alpha} \\
&= \vec{R} \times \sum_{\alpha} \vec{P}_{\alpha} + \sum_{\alpha} \vec{r}'_{\alpha} \times \vec{P}_{\alpha} \\
&= \vec{R} \times \vec{P}_{\text{tot}} + \sum_{\alpha} \vec{r}'_{\alpha} \times m_{\alpha} \dot{\vec{r}}_{\alpha} \\
&= \vec{R} \times \vec{P}_{\text{tot}} + \sum_{\alpha} \vec{r}'_{\alpha} \times m_{\alpha} (\dot{\vec{R}} + \dot{\vec{r}}'_{\alpha}) \\
&= \vec{R} \times \vec{P}_{\text{tot}} + \left(\sum_{\alpha} m_{\alpha} \vec{r}'_{\alpha} \right) \times \dot{\vec{R}} \\
&\quad + \sum_{\alpha} \vec{r}'_{\alpha} \times m_{\alpha} \dot{\vec{r}}'_{\alpha}.
\end{aligned}
$$

The middle term vanishes. Thus

$$
\boxed{\vec{L}_{\text{tot}} = \vec{L}_{\text{orb}} + \vec{L}_{\text{spin}}},
$$

$$
\vec{L}_{\text{orb}} = \vec{R} \times \vec{P}_{\text{tot}},
$$

$$
\vec{L}_{\text{spin}} = \sum_{\alpha} \vec{r}'_{\alpha} \times m_{\alpha} \dot{\vec{r}}'_{\alpha}.
$$

Here "spin" denotes classical internal angular momentum about the center of mass; it does not require quantum intrinsic spin.

11.3.2 External torque and cancellation of internal torques

First,

$$
\begin{aligned}
\frac{d\vec{L}_{\text{orb}}}{dt} &= \underbrace{\dot{\vec{R}} \times \vec{P}_{\text{tot}}}_{=0} + \vec{R} \times \dot{\vec{P}}_{\text{tot}} \\
&= \vec{R} \times \vec{F}^{\,\text{ext}}.
\end{aligned}
$$

For the total angular momentum,

$$
\begin{aligned}
\frac{d\vec{L}_{\text{tot}}}{dt} &= \sum_{\alpha} \vec{r}_{\alpha} \times m_{\alpha} \ddot{\vec{r}}_{\alpha} \\
&= \sum_{\alpha} \vec{r}_{\alpha} \times \vec{F}_{\alpha}^{\text{ext}} + \sum_{\alpha \neq \beta} \vec{r}_{\alpha} \times \vec{F}_{\alpha\beta}.
\end{aligned}
$$

Equal and opposite pair forces, $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$, give

$$
\begin{aligned}
\vec{\tau}_{\text{int}} &= \sum_{\alpha < \beta} (\vec{r}_{\alpha} \times \vec{F}_{\alpha\beta} + \vec{r}_{\beta} \times \vec{F}_{\beta\alpha}) \\
&= \sum_{\alpha < \beta} (\vec{r}_{\alpha} - \vec{r}_{\beta}) \times \vec{F}_{\alpha\beta}.
\end{aligned}
$$

If the internal pair forces also act along the line joining each pair, every term vanishes. Equal and opposite forces alone are insufficient for this cancellation. Under these assumptions,

$$\begin{aligned}\frac{d\vec{L}_{\text{tot}}}{dt} &= \sum_{\alpha} \vec{r}_{\alpha} \times \vec{F}_{\alpha}^{\text{ext}} \\ &= \vec{R} \times \vec{F}^{\text{ext}} + \sum_{\alpha} \vec{r}'_{\alpha} \times \vec{F}_{\alpha}^{\text{ext}}.\end{aligned}$$

Subtracting the orbital balance gives

$$\boxed{\frac{d\vec{L}_{\text{spin}}}{dt} = \sum_{\alpha} \vec{r}'_{\alpha} \times \vec{F}_{\alpha}^{\text{ext}}}.$$

The right-hand side is the external torque about the center of mass.

Angular momentum in quantum mechanics

In quantum mechanics, orbital and intrinsic spin angular momenta are quantized. Orbital quantum numbers are integers; spin quantum numbers may be integers or half-integers. For example, a spin-1/2 particle has spin-component values $\pm\hbar/2$ along a chosen axis. In general, the angular-momentum magnitude is $\sqrt{j(j+1)}\hbar$, while a measured component is $m_j\hbar$.

11.4 Proton scattering by a heavy nucleus

Treat the nucleus as fixed and neglect radiation. Let its charge be $Q > 0$, the proton charge be $e > 0$, and write the repulsive Coulomb potential in SI units as

$$U(r) = \frac{\kappa}{r}, \qquad \kappa = \frac{Qe}{4\pi\epsilon_0}.$$

The orbit is a hyperbola (双曲线) with the nucleus at a focus. Its asymptotes (渐近线) intersect at the geometric center of the hyperbola.

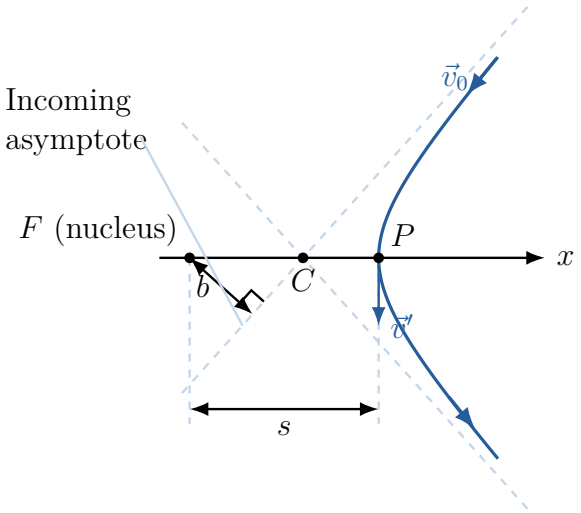


Fig. 3: Repulsive Coulomb scattering. The nucleus F is a focus; the asymptotes intersect at the geometric center C. The impact parameter b is perpendicular to the incoming asymptote, and s reaches the nearest point P.

The perpendicular distance b from the nucleus to the incoming asymptote is called the impact parameter (碰撞参数). Given b and the incident speed v_0 at large distance, what is the smallest separation s between the proton and the nucleus? At closest approach, its velocity is perpendicular to its radius vector and has magnitude v'. Angular momentum conservation gives

$$mv_0 b = mv' s,$$

$$v' = v_0 \frac{b}{s}.$$

Energy conservation then gives

$$\begin{aligned} \frac{1}{2}mv_0^2 &= \frac{1}{2}mv'^2 + \frac{\kappa}{s} \\ &= \frac{1}{2}mv_0^2\frac{b^2}{s^2} + \frac{\kappa}{s}. \end{aligned}$$

Equivalently,

$$\frac{\kappa}{s} = \frac{1}{2}mv_0^2\left[1 - \left(\frac{b}{s}\right)^2\right].$$

Writing

$$a = \frac{\kappa}{mv_0^2},$$

solve the quadratic:

$$s^2 - 2as - b^2 = 0,$$

$$\boxed{s = a + \sqrt{a^2 + b^2}}.$$

11.5 Effective potential and rotational support

11.5.1 Radial motion in a gravitational field

A rotating cloud can contract differently along and perpendicular to its angular momentum. To isolate the centrifugal effect, first consider one particle of mass m moving in the field of a fixed mass M. In its orbital plane,

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta,$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta.$$

The cross terms cancel on adding the squares:

$$\begin{aligned} E_k &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2). \end{aligned}$$

The total energy is

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}.$$

Choose the positive z direction along $\vec{L}$. Then

$$\begin{aligned} L = L_z &= m(x\dot{y} - y\dot{x}) \\ &= mr\cos\theta\big(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\big) \\ &\quad - mr\sin\theta\big(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\big) \\ &= mr^2\dot{\theta}, \\ \dot{\theta} &= \frac{L}{mr^2}. \end{aligned}$$

Substituting this result into the conserved energy gives

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r),$$

$$U_{\text{eff}}(r) = \underbrace{\frac{L^2}{2mr^2}}_{\text{centrifugal term}} - \underbrace{\frac{GMm}{r}}_{\text{gravitational term}}.$$

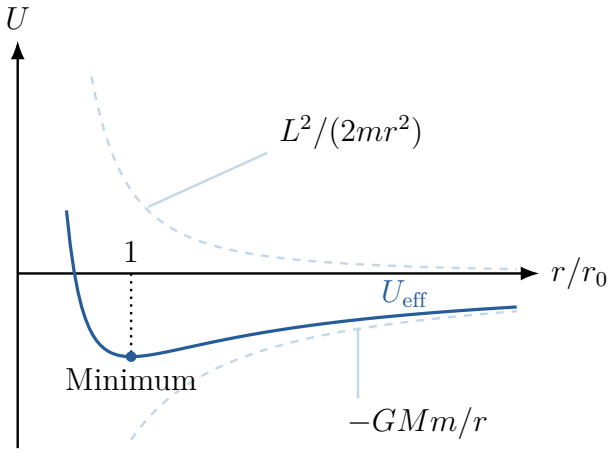


Fig. 4: The centrifugal term rises as $1/r^2$, while gravity contributes $-GMm/r$. Their sum has a minimum at r_0 for $L \neq 0$.

The circular-orbit radius at fixed L satisfies

$$0 = U'_{\text{eff}}(r_0) = -\frac{L^2}{mr_0^3} + \frac{GMm}{r_0^2},$$

$$r_0 = \frac{L^2}{GMm^2}.$$

At that radius,

$$U''_{\text{eff}}(r_0) = \frac{GMm}{r_0^3} > 0,$$

so the circular orbit is stable against small radial perturbations. A general orbit can pass inside r_0: its turning radii are determined by $E = U_{\text{eff}}(r)$, rather than by the minimum alone.

11.5.2 Why a galactic disk and a dark matter halo?

Compare contraction toward the rotation axis with settling along it. The effective potential distinguishes these two directions.

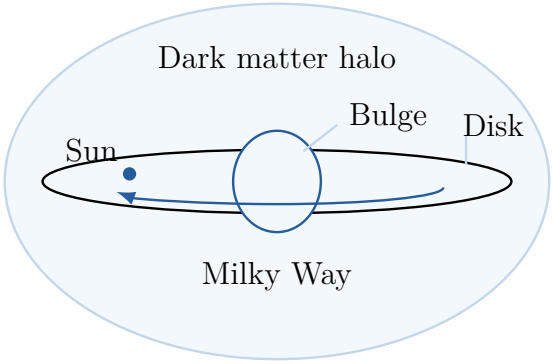


Fig. 5: Schematic Milky Way components: disk, central bulge (核球), and an extended dark matter halo. The Sun lies in the disk; the arrow indicates disk rotation. Sizes and halo shape are illustrative.

From the effective potential to a disk

Both gas and dark matter feel gravity and can carry angular momentum. An isolated system conserves total angular momentum. In the central-mass approximation above, choose z along the net angular momentum and let R be the distance from this axis. For a particle with fixed axial component L_z,

$$U_{\text{eff}}(R, z) = \frac{L_z^2}{2mR^2} - \frac{GMm}{\sqrt{R^2 + z^2}}.$$

Radial contraction (R). In the plane $z = 0$, the growing centrifugal term limits contraction toward the axis. For $L_z \neq 0$, the potential has the same nonzero-radius minimum as above:

$$R_0 = \frac{L_z^2}{GMm^2} > 0.$$

Vertical settling (z). At fixed R, there is no additional centrifugal term opposing vertical contraction. Gravity pulls matter toward $z = 0$; without damping, particles cross the plane and oscillate.

Gas: collisions and shocks convert disordered motion into heat; escaping photons carry that energy away. Cooling damps vertical motion while substantial net angular momentum is retained, so gas settles into a rotating disk. Residual pressure gives the disk a finite thickness. Stars formed from this gas inherit its disk motion.[1]

Dark matter: in the standard non-dissipative picture, random orbital motions persist and support an extended halo. Elastic self-scattering alone redistributes energy without providing radiative cooling.[2]

See also *Dark matter: from orbital motion to resonance.*[3]

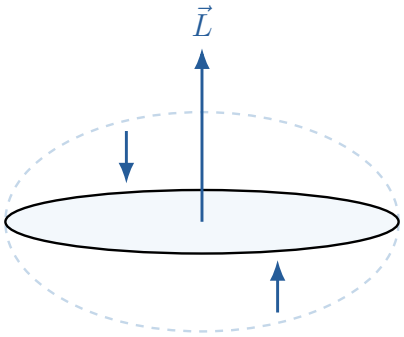


Fig. 6: Schematic flattening of a rotating cloud. Rotation provides support perpendicular to the angular momentum axis.

References

[1] S. M. Fall and G. Efstathiou, *Formation and rotation of disc galaxies with haloes*, Mon. Not. R. Astron. Soc. **193** (1980), 189–206. https://doi.org/10.1093/mnras/193.2.189

[2] S. Tulin and H.-B. Yu, *Dark matter self-interactions and small scale structure*, Phys. Rep. **730** (2018), 1–57, Secs. I and III. https://arxiv.org/abs/1705.02358

[3] General Physics I (2026), supplementary reading, *Dark matter: from orbital motion to resonance*. Course reading PDF.