

Lecture 10: Conservation laws: momentum conservation

- Newton's third law
- The symmetry aspect of momentum conservation
- Collisions
- The Tsiolkovsky equation

10.1 Newton's third law and its limitations

Action equals reaction. Let $\vec{F}_{12}$ denote the force on object 1 exerted by object 2.

$$\vec{F}_{12} = -\vec{F}_{21}.$$

Problem: Newton's third law assumes instantaneous action and reaction, which is inconsistent with relativity. For two moving charges in electromagnetism, even the force directions need not be parallel.

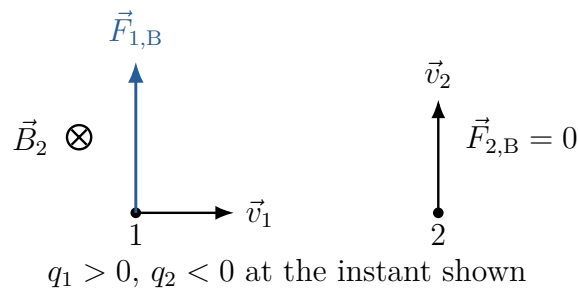


Fig. 1: The magnetic force on particle 1 is upward; the magnetic force on particle 2 vanishes at the instant shown.

The key to understanding these limitations is the modern perspective of **fields**.

Charge (mass) generates fields. Changes in the fields propagate at the speed of light, and the fields act on charge (mass). Nevertheless, within the Newtonian models considered here, at speeds much less than the speed of light, we assume Newton's third law to be valid.

Consider two objects 1 and 2, with $\vec{F}_{12} = -\vec{F}_{21}$ and no external force.

$$\begin{aligned}\frac{d\vec{P}_1}{dt} &= \vec{F}_{12}, \\ \frac{d\vec{P}_2}{dt} &= \vec{F}_{21}, \\ \frac{d}{dt}(\vec{P}_1 + \vec{P}_2) &= 0.\end{aligned}$$

10.2 Symmetry and Galilean transformations (perspective)

10.2.1 Equal masses

Consider a mass $2m$, initially at rest. Imagine an explosion that splits it into two parts of equal mass. The system has reflection symmetry, so the velocities are equal in magnitude and opposite in direction.

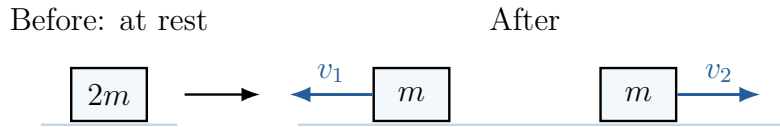


Fig. 2: Symmetric separation of a stationary body: the two outgoing speeds are equal.

Hence, the total momentum before and after the explosion is zero.

Consider two equal masses with velocities v_1 and v_2 along one line. They collide and stick together. What is their final velocity?

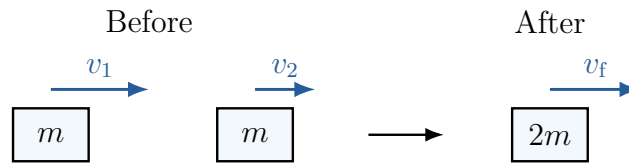


Fig. 3: Equal masses before and after a perfectly inelastic collision.

We switch to a reference frame F' moving at $V = (v_1 + v_2)/2$. In this frame, the two masses have opposite velocities,

$$v'_1 = \frac{v_1 - v_2}{2},$$

$$v'_2 = -\frac{v_1 - v_2}{2}.$$

They therefore exhibit reflection symmetry. If they stick together, the final velocity in this frame has to be zero. Transforming back to the laboratory frame gives

$$v_f = \frac{v_1 + v_2}{2},$$

$$2mv_f = mv_1 + mv_2.$$

10.2.2 Unequal masses

Consider a mass $3m$, initially at rest. It splits into two unequal masses, m and $2m$. What is the relation between v and v' ?

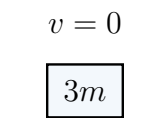


Fig. 4: Initial stationary body of mass $3m$.

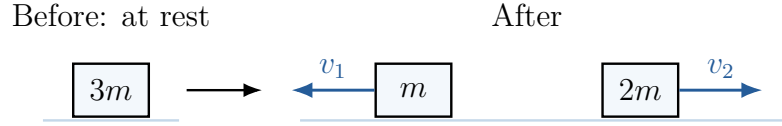


Fig. 5: The mass $3m$ separates into m and $2m$, with opposite velocities $v_1 = v$ and $v_2 = v'$.

We first view $3m$ as two masses, $2m$ and m . The mass $2m$ then splits into two equal masses. Recombine one moving mass m with the stationary mass m , as shown below.

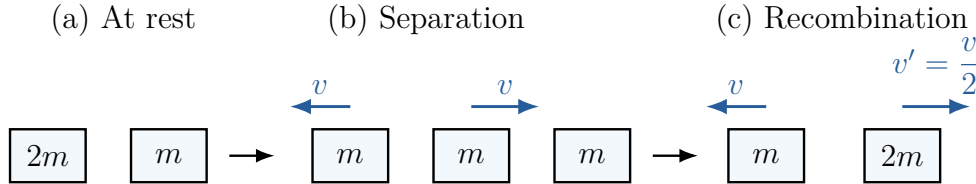


Fig. 6: Equal-mass construction of the 1 : 2 mass ratio. The two units that recombine have velocities v and zero, so their common velocity is $v/2$.

From symmetry, we conclude that when the mass ratio is $m_1 : m_2 = 1 : 2$, the speed ratio is $v_1 : v_2 = 2 : 1$, with opposite directions. Hence, the total momentum remains zero.

Reverse the separation picture and apply a Galilean transformation. For the corresponding collision with mass ratio 1 : 2, if the two masses stick together, momentum conservation gives

$$3mv_f = mv_1 + 2mv_2,$$

$$v_f = \frac{v_1 + 2v_2}{3}.$$

Using a similar method, if a mass splits into two parts with $m_1 : m_2 = p : q$, where p and q are positive integers, their velocities are opposite in direction and their speed ratio is $v_1 : v_2 = q : p$.

$$m_1\vec{v}_1 + m_2\vec{v}_2 = 0,$$

$$\frac{|\vec{v}_1|}{|\vec{v}_2|} = \frac{q}{p}.$$

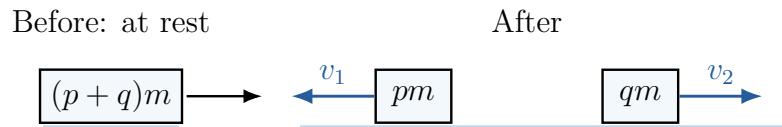


Fig. 7: Separation into masses pm and qm with opposite momenta. Arrow lengths indicate direction only.

10.3 A system of N interacting point particles

For N interacting point particles with constant masses,

$$\begin{cases} m_1 \ddot{\vec{r}}_1 = \vec{F}_1^{\text{ext}} + \vec{F}_{12} + \vec{F}_{13} + \cdots + \vec{F}_{1N}, \\ m_2 \ddot{\vec{r}}_2 = \vec{F}_2^{\text{ext}} + \vec{F}_{21} + \vec{F}_{23} + \cdots + \vec{F}_{2N}, \\ \vdots \\ m_N \ddot{\vec{r}}_N = \vec{F}_N^{\text{ext}} + \vec{F}_{N1} + \cdots + \vec{F}_{N,N-1}. \end{cases}$$

Adding all equations gives

$$\begin{aligned} \sum_{i=1}^N m_i \ddot{\vec{r}}_i &= \frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\vec{r}}_i \right) \\ &= \sum_{i=1}^N \vec{F}_i^{\text{ext}} + \sum_{i \neq j} \vec{F}_{ij}. \end{aligned}$$

Since $\vec{F}_{ij} + \vec{F}_{ji} = 0$,

$$\frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\vec{r}}_i \right) = \vec{F}_{\text{tot}}^{\text{ext}}.$$

With $M = \sum_{i=1}^N m_i$,

$$\begin{aligned} \vec{P}_{\text{tot}} &= \sum_{i=1}^N m_i \dot{\vec{r}}_i \\ &= M \frac{d}{dt} \left(\sum_{i=1}^N \frac{m_i}{M} \vec{r}_i \right) \\ &= M \dot{\vec{R}}, \end{aligned}$$

where the center-of-mass position (质心位置) is

$$\vec{R} = \sum_{i=1}^N \frac{m_i}{M} \vec{r}_i.$$

This explains why the point-mass description is valid even when applied to an extended object. For translational motion, an extended object can be treated as if all its mass were concentrated at its center of mass.

10.4 Equal-mass elastic collisions

Consider an elastic collision between two equal masses m , with particle 2 initially at rest. Primes denote the final velocities.

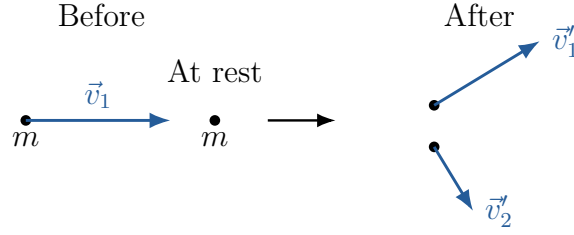


Fig. 8: An incident particle strikes an equal mass initially at rest.

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 \quad (\text{energy conservation}),$$

$$m\vec{v}_1 = m\vec{v}_1' + m\vec{v}_2' \quad (\text{momentum conservation}).$$

The momentum equation gives

$$v_1^2 = v_1'^2 + v_2'^2 + 2\vec{v}_1' \cdot \vec{v}_2'.$$

Energy conservation gives

$$v_1^2 = v_1'^2 + v_2'^2.$$

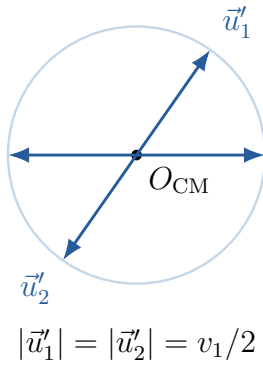
Comparing the two equations,

$$\boxed{\vec{v}_1' \cdot \vec{v}_2' = 0}.$$

Thus the two outgoing velocities are perpendicular, or one of them is zero. Let $\vec{u}_i$ denote the final velocities in the center-of-mass frame, which moves at $\vec{V}_{\text{CM}}$ relative to the laboratory. Then

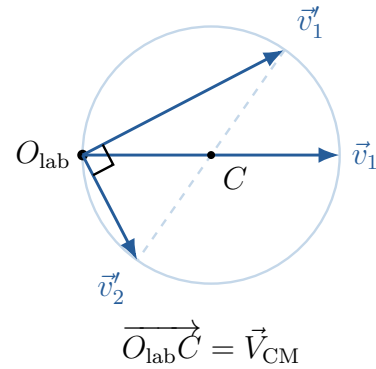
$$\vec{V}_{\text{CM}} = \frac{\vec{v}_1}{2},$$

$$\vec{v}_i' = \vec{V}_{\text{CM}} + \vec{u}_i', \quad i = 1, 2.$$



$$|\vec{u}_1'| = |\vec{u}_2'| = v_1/2$$

Fig. 9: Center-of-mass frame: the two velocities remain opposite, with unchanged magnitudes in an elastic collision. Horizontal arrows denote the incoming velocities.



$$\overrightarrow{O_{\text{lab}}C} = \vec{V}_{\text{CM}}$$

Fig. 10: Adding the center-of-mass velocity gives the laboratory velocity circle. The outgoing velocity vectors subtend a right angle across its diameter.

10.5 Momentum conservation in dark matter collisions

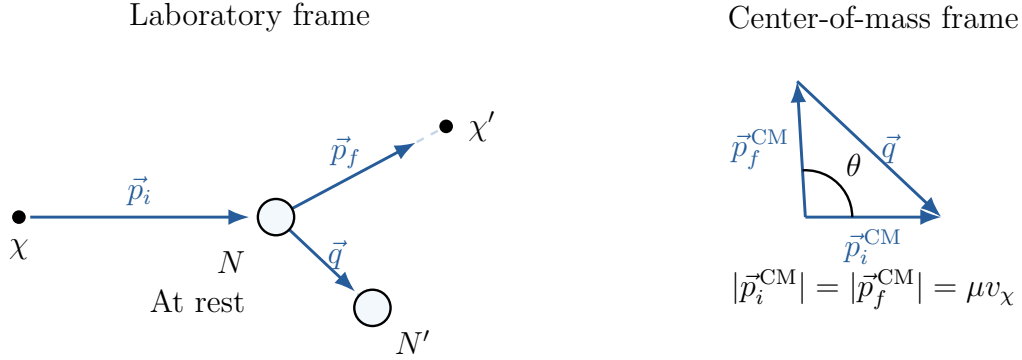


Fig. 11: An incident dark matter particle χ scatters from a stationary nucleus N in the laboratory (left), producing χ' and a recoiling nucleus N' . The center-of-mass momentum triangle (right) defines the scattering angle θ used below; $\vec{q} = \vec{p}_i^{\text{CM}} - \vec{p}_f^{\text{CM}}$.

In astrophysics, momentum conservation is one of the key tools for understanding how dark matter (DM) behaves on microscopic and galactic scales. In the nonrelativistic model considered here, dark matter has no direct electromagnetic interaction, but its dynamics still obey Newton's law of action and reaction — total momentum is conserved in an isolated elastic or self-interacting process.

When a dark matter particle χ collides elastically with a nucleus N ,

$$\chi + N \longrightarrow \chi' + N',$$

momentum conservation gives

$$m_\chi \vec{v}_\chi + m_N \vec{v}_N = m_\chi \vec{v}_{\chi'} + m_N \vec{v}_{N'}.$$

In the laboratory frame, the nucleus is initially at rest ($\vec{v}_N = 0$), so it recoils with momentum $\vec{q}$. With the reduced mass (约化质量)

$$\mu = \frac{m_\chi m_N}{m_\chi + m_N},$$

the momentum transfer has magnitude

$$|\vec{p}_{N'}| = q = 2\mu v_\chi \sin \frac{\theta}{2},$$

where θ is the scattering angle in the center-of-mass frame. The recoil energy is

$$E_R = \frac{q^2}{2m_N},$$

Galactic dark matter velocities can produce recoil energies in the keV range, depending on the dark matter and nuclear masses.

Even in the absence of direct electromagnetic interaction, total momentum is conserved in these dark matter collisions. This principle provides a theoretical basis for laboratory-scale experiments.

10.6 The Tsiolkovsky rocket equation (1903)

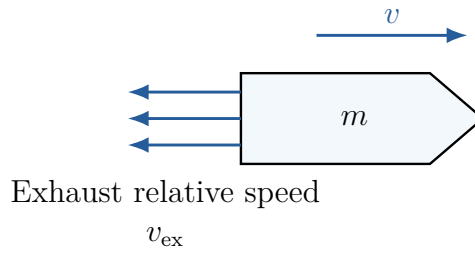


Fig. 12: The rocket moves to the right; exhaust is expelled backward relative to the rocket.

The effective exhaust speed v_{ex} is the speed of the exhaust gas relative to the rocket. Take v_{ex} to be constant and neglect external forces.

At time t ,

$$P(t) = mv.$$

At time $t + dt$, the rocket mass is $m + dm$, where $dm < 0$, and the fuel burned has mass $-dm$. The total momentum is

$$\begin{aligned} P(t + dt) &= (m + dm)(v + dv) + (-dm)(v - v_{\text{ex}}) \\ &\simeq mv + m dv + v_{\text{ex}} dm. \end{aligned}$$

Since $P(t + dt) = P(t)$,

$$m dv = -v_{\text{ex}} dm,$$

$$dv = -v_{\text{ex}} \frac{dm}{m}.$$

Hence,

$$\begin{aligned} \int_{v_0}^v dv &= -v_{\text{ex}} \int_{m_0}^m \frac{dm}{m}, \\ v - v_0 &= v_{\text{ex}} \ln \frac{m_0}{m}. \end{aligned}$$

If 90% of the rocket's initial mass is fuel, then $m_0/m = 10$, so

$$\Delta v = v_{\text{ex}} \ln 10 \simeq 2.3v_{\text{ex}}.$$