

Lecture 9: Conservation law one: energy conservation (part II)

Energy as a solver. The conservation law is not just a statement of bookkeeping. It turns difficult second-order motion into geometry on an energy curve.

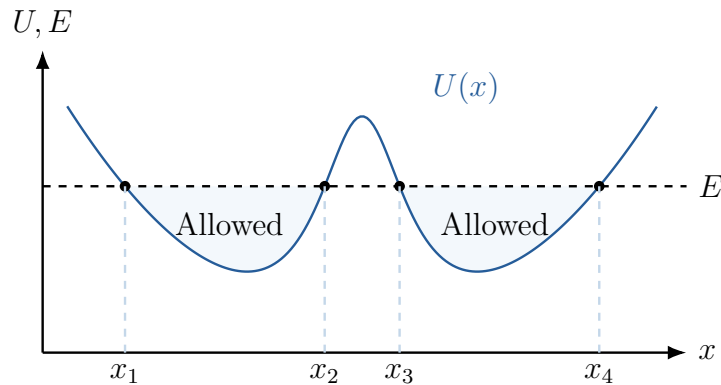


Fig. 1: The shaded regions satisfy $U(x) \leq E$; intersections of the potential with the dashed energy line are turning points.

9.1 Several forces

If all forces come from time-independent potentials, then

$$\vec{F}_1 = -\nabla U_1, \quad \vec{F}_2 = -\nabla U_2, \quad \dots$$

and

$$E = E_K + U_1(\vec{r}) + U_2(\vec{r}) + \dots \quad \text{is conserved}$$

9.1.1 A spring in a gravitational field

Take z upward, with $z = 0$ at the endpoint of the unstretched spring. Both gravity and the spring are conservative:

$$E = E_K + mgz + \frac{1}{2}kz^2$$

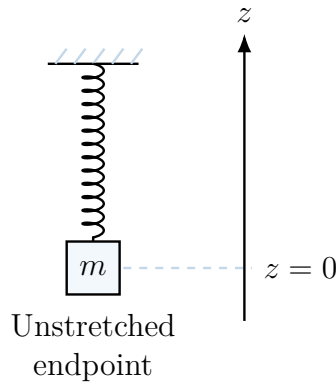


Fig. 2: Vertical spring in uniform gravity. The upward coordinate z is zero at the unstretched spring endpoint.

9.1.2 Non-conservative forces

$$\begin{aligned}\Delta E_K &= W_{\text{con}} + W_{\text{ncon}} = -\Delta U + W_{\text{ncon}} \\ \Rightarrow \Delta(E_K + U) &= W_{\text{ncon}}\end{aligned}$$

Example: a rough incline.

$$\Delta(E_K + U) = -fd = -\mu mg \cos \theta d$$

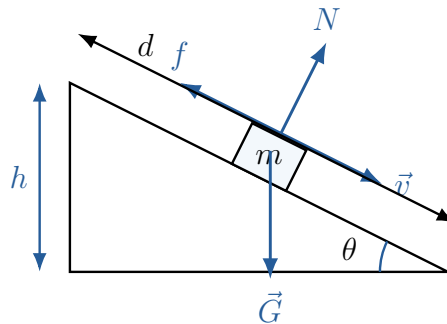


Fig. 3: A block sliding down a rough incline: friction opposes the motion, the normal force is perpendicular to it, and $h = d \sin \theta$.

$$\begin{aligned}E_{K,i} &= 0, & U_i &= mgh = mgd \sin \theta, \\ U_f &= 0, \\ E_{K,f} &= mgd \sin \theta - \mu mgd \cos \theta \\ &= \frac{1}{2}mv_f^2, \\ v_f &= \sqrt{2gd(\sin \theta - \mu \cos \theta)}.\end{aligned}$$

Here μ is the kinetic friction coefficient. The block must already be sliding, or the slope must overcome static friction; starting from rest and accelerating downhill also requires $\sin \theta > \mu \cos \theta$. Mechanical energy lost through friction becomes internal energy of the block and incline.

9.2 One-dimensional motion

If F_x is only coordinate-dependent, then F_x is conservative. This is because any closed loop in 1D has to come back along the same path $\int_1^2 dx F_x + \int_2^1 dx F_x = 0$

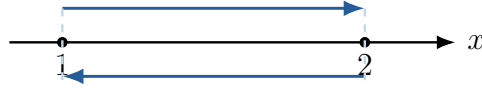


Fig. 4: In one dimension, a return path retraces the same interval with opposite orientation.

Then the potential energy $U(x)$ can be simply integrated as

$$U(x) = - \int_{x_0}^x F_x(x') dx'$$

Choosing another reference point changes $U(x)$ only by an additive constant; here $U(x_0) = 0$.

Does moving the zero of energy change the motion? Replace $U(x)$ by $\tilde{U}(x) = U(x) + C$ and the same trajectory has $\tilde{E} = E + C$. Then

$$-\frac{d\tilde{U}}{dx} = -\frac{dU}{dx}, \quad \tilde{E} - \tilde{U} = E - U.$$

The force, turning points, speeds, and period are unchanged. Shifting the potential and keeping the numerical total energy fixed would describe a different physical trajectory.

9.2.1 A diatomic molecule

(1) $U_{\min} < E < 0$ — bound motion

At b and $d \Rightarrow E_K = 0$, no kinetic energy, turning points.

At c , $\frac{\partial U}{\partial r} = 0$, $\frac{\partial^2 U}{\partial r^2} > 0$. $\Rightarrow c$ is a stable equilibrium point

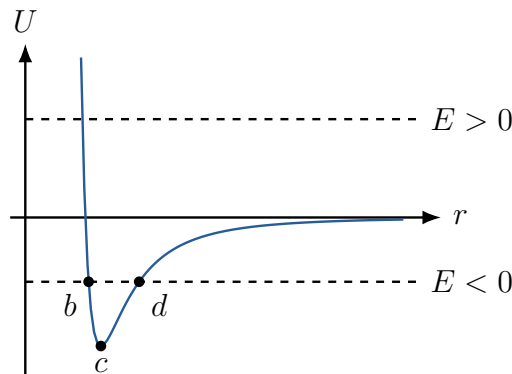


Fig. 5: A molecular potential: b, d are turning points for the indicated negative energy, and c is a stable equilibrium.

(2) $E > 0$ — unbound scattering motion

For the relative vibration of two atoms, the mass parameter is the reduced mass $\mu = m_1 m_2 / (m_1 + m_2)$; rotational motion is omitted here.

Why do so many systems behave like springs?

Near a smooth nondegenerate minimum at r_0 (point c), let $\xi = r - r_0$. Since $U'(r_0) = 0$,

$$U(r) = U(r_0) + \frac{1}{2}U''(r_0)\xi^2 + O(\xi^3), \quad U''(r_0) > 0.$$

To leading order,

$$\mu\ddot{\xi} = -U''(r_0)\xi, \quad \omega_0^2 = \frac{U''(r_0)}{\mu}.$$

The curvature supplies an effective spring constant. This is the natural frequency entering the driven-oscillator and resonance discussion in Lecture 5; the exact response peak also depends on damping and on the measured response. Large oscillations probe the higher-order terms and need not have an amplitude-independent period.

9.2.2 Recovering time from the energy curve

We can solve one-dimensional motion by a single integral:

$$E_K = \frac{1}{2}m\dot{x}^2 = E - U(x),$$

$$\dot{x}(x) = \pm \sqrt{\frac{2[E - U(x)]}{m}}.$$

The sign of $\dot{x}(x)$ distinguishes right-moving and left-moving segments.

On each segment where the velocity has a fixed sign,

$$dt = \frac{dx}{\dot{x}(x)}, \quad t_f - t_i = \int_{x_i}^{x_f} \frac{dx}{\dot{x}(x)}.$$

Suppose $\dot{x}$ is positive, we have

$$t_f - t_i = \sqrt{\frac{m}{2}} \int_{x_i}^{x_f} \frac{dx'}{\sqrt{E - U(x')}}.$$

$\dot{x}$ can change directions at turning points, and we can treat by dividing the motion into different regions. In each region, the sign of $\dot{x}$ is fixed, and we add the time of each region together.

Example 1: free fall. Measure z downward and release the particle from rest at $z_{\text{in}} = 0$, $v_{\text{in}} = 0$. With this reference choice,

$$U(z) = -mgz, \quad E = 0.$$

$$\Rightarrow \dot{z}(z) = \sqrt{\frac{2}{m}} \sqrt{E - U(z)} = \sqrt{2gz}$$

$$t = \int_0^z \frac{dz'}{\dot{z}(z')} = \int_0^z \frac{dz'}{\sqrt{2gz'}} = \sqrt{\frac{2z}{g}} \Rightarrow z = \frac{1}{2}gt^2$$

Example 2: harmonic oscillator.

$U = \frac{1}{2}kx^2$ with energy E .

The turning points at $\pm A$, with

$$\frac{1}{2}kA^2 = E.$$

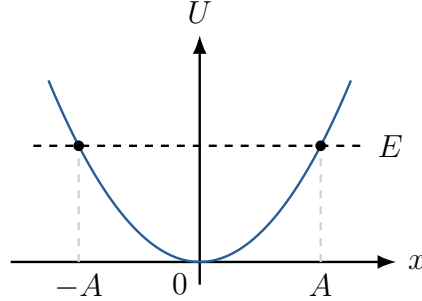


Fig. 6: Harmonic potential: the intersections with the energy line give the turning points $x = \pm A$.

Start at $t = 0$ with $x = A$ and $\dot{x} = 0$. The mass first moves left, reaching $x = 0$ at $t = T/4$:

$$\dot{x} = -\sqrt{\frac{2}{m} \left(E - \frac{1}{2}kx^2 \right)}.$$

Using $E = kA^2/2$ and then $y = x/A$,

$$\begin{aligned} \frac{T}{4} &= \int_A^0 \frac{dx}{\dot{x}} \\ &= \sqrt{\frac{m}{2}} \int_0^A \frac{dx}{\sqrt{E - \frac{1}{2}kx^2}} \\ &= \sqrt{\frac{m}{k}} \int_0^A \frac{dx}{\sqrt{A^2 - x^2}} \\ &= \frac{1}{\omega_0} \int_0^1 \frac{dy}{\sqrt{1 - y^2}} \\ &= \frac{\arcsin y}{\omega_0} \Big|_0^1 = \frac{\pi}{2\omega_0}. \end{aligned}$$

Hence

$$\boxed{T = \frac{2\pi}{\omega_0}}, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

9.3 Several objects: Atwood machine with constraints

Two masses are suspended by a massless inextensible string over a fixed, frictionless, massless pulley. Let s_1, s_2 denote upward displacements of the masses, and let the tension magnitude be T :

$$\Delta(E_{K_1} + U_1) = W_1^{ten}$$

$$\Delta(E_{K_2} + U_2) = W_2^{ten}$$

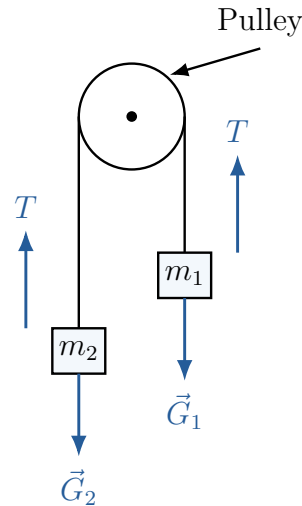


Fig. 7: Ideal Atwood machine: equal tensions and opposite displacements give zero total work by tension.

The tensions are equal, and the fixed string length implies $ds_1 + ds_2 = 0$.

The inextensible string stores no elastic energy, so

$$\begin{aligned} W_1^{\text{ten}} + W_2^{\text{ten}} &= \int T ds_1 + \int T ds_2 \\ &= \int T (ds_1 + ds_2) = 0 \\ \Rightarrow \Delta(E_{K_1} + E_{K_2} + U_1 + U_2) &= 0 \end{aligned}$$

In general, if a system contains several particles, with constraints, and if the constraining force does not do the total work on the system, they can be neglected in the total energy.

9.3.1 Constraints and energy bookkeeping

Energy conservation is most useful when we choose the system boundary carefully. Ideal constraints, such as the fixed-pulley string above or a stationary frictionless surface, can exert large forces while doing zero total work on the system. Then their detailed values need not appear in the final energy equation. Non-conservative forces are different: they do not disappear; their work is the change of mechanical energy,

$$\Delta(E_K + U) = W_{\text{nonconservative}}.$$

This is why energy methods can solve constrained systems quickly, but still keep track of friction, drag, or external driving.

9.4 Energy of two interacting particles

Let $\vec{F}_{12}$ be the force on particle 1 due to particle 2, and define $\vec{r} = \vec{r}_1 - \vec{r}_2$. Assume a time-independent conservative interaction that depends only on this relative displacement:

$$U_{12} = U_{12}(\vec{r}_1 - \vec{r}_2).$$

A common translation of both particles leaves this potential unchanged. Its two gradients give equal and opposite forces.

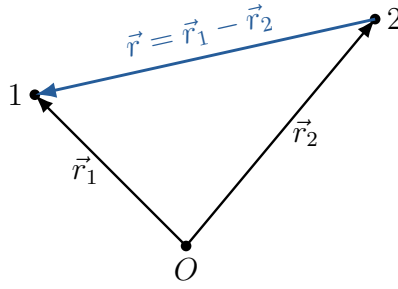


Fig. 8: The relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ points from particle 2 to particle 1.

With particle 2 fixed, the conservative force on particle 1 satisfies

$$\oint \vec{F}_{12} \cdot d\vec{r}_1 = 0, \quad \vec{F}_{12} = -\nabla_{\vec{r}_1} U_{12}.$$

Why do the two gradients have opposite signs?

Write $\vec{r}_i = (x_i, y_i, z_i)$ in a common fixed Cartesian basis. The gradient (梯度) with respect to the position of particle i is

$$\nabla_{\vec{r}_i} U_{12} = \hat{e}_x \frac{\partial U_{12}}{\partial x_i} + \hat{e}_y \frac{\partial U_{12}}{\partial y_i} + \hat{e}_z \frac{\partial U_{12}}{\partial z_i}, \quad i = 1, 2.$$

In $\nabla_{\vec{r}_1} U_{12}$, $\vec{r}_2$ is held fixed; in $\nabla_{\vec{r}_2} U_{12}$, $\vec{r}_1$ is held fixed. Each component measures the change of potential per unit displacement along that coordinate.

For the same potential written as $U_{12}(\vec{r})$, with relative coordinates $\vec{r} = (x, y, z) = \vec{r}_1 - \vec{r}_2$, define

$$\nabla_{\vec{r}} U_{12} = \hat{e}_x \frac{\partial U_{12}}{\partial x} + \hat{e}_y \frac{\partial U_{12}}{\partial y} + \hat{e}_z \frac{\partial U_{12}}{\partial z}.$$

Moving particle 1 by $+d\vec{r}_1$ increases $\vec{r}$ by the same vector; moving particle 2 by $+d\vec{r}_2$ changes $\vec{r}$ by $-d\vec{r}_2$. For example, $x = x_1 - x_2$ gives

$$\frac{\partial U_{12}}{\partial x_1} = \frac{\partial U_{12}}{\partial x} \frac{\partial (x_1 - x_2)}{\partial x_1} = +\frac{\partial U_{12}}{\partial x},$$

$$\frac{\partial U_{12}}{\partial x_2} = \frac{\partial U_{12}}{\partial x} \frac{\partial (x_1 - x_2)}{\partial x_2} = -\frac{\partial U_{12}}{\partial x}.$$

The y and z components follow in the same way, so

$$\nabla_{\vec{r}_1} U_{12} = \nabla_{\vec{r}} U_{12}, \quad \nabla_{\vec{r}_2} U_{12} = -\nabla_{\vec{r}} U_{12}.$$

Thus the same interaction potential gives

$$\vec{F}_{12} = -\nabla_{\vec{r}_1} U_{12} = -\nabla_{\vec{r}} U_{12},$$

$$\vec{F}_{21} = -\nabla_{\vec{r}_2} U_{12} = +\nabla_{\vec{r}} U_{12} = -\vec{F}_{12}.$$

This is Newton's third law for the assumed interaction model.

If the internal forces cancel, why can total kinetic energy change? The two forces act through different displacements. Applying the work–energy theorem to each particle and adding gives

$$\begin{aligned} d(E_{K,1} + E_{K,2}) &= \vec{F}_{12} \cdot d\vec{r}_1 + \vec{F}_{21} \cdot d\vec{r}_2 \\ &= \vec{F}_{12} \cdot (d\vec{r}_1 - d\vec{r}_2) \\ &= -\nabla_{\vec{r}} U_{12} \cdot d\vec{r} \\ &= -dU_{12}. \end{aligned}$$

$$\boxed{d(E_{K,1} + E_{K,2} + U_{12}) = 0.}$$

The relative displacement determines the total internal work; the interaction potential is counted once for the pair.

In principle, we can also include the external conservative forces on 1 and 2, and introduce potentials. U_1^{ex} and U_2^{ex} , then

$$E = E_{K,1} + E_{K,2} + U_1^{\text{ext}} + U_2^{\text{ext}} + U_{12}.$$

For n particles with time-independent external and pair potentials, assuming all interactions are described by these terms,

$$\begin{aligned} E &= E_{K,1} + \cdots + E_{K,n} + U_1^{\text{ext}} + \cdots + U_n^{\text{ext}} \\ &\quad + U_{12} + U_{13} + \cdots + U_{n-1,n} \\ &= \underbrace{\sum_{i=1}^n (E_{K,i} + U_i^{\text{ext}})}_{\text{single-particle terms}} + \underbrace{\sum_{i < j} U_{ij}}_{\text{each interacting pair counted once}}. \end{aligned}$$

9.5 Time-dependent potential energy

If $\vec{F}(\vec{r}, t)$ has zero work around every closed spatial loop at fixed t , it can be written as $\vec{F} = -\nabla U(\vec{r}, t)$. Explicit time dependence allows the mechanical energy $E = E_K + U$ to change.

9.5.1 A slowly changing charge

Consider a source near the origin whose charge $Q(t)$ is controlled by an external circuit. A test particle of fixed charge q is at $\vec{r}$, with $r = |\vec{r}|$. In a quasistatic (准静态) Coulomb model outside the source, choosing $U(\infty, t) = 0$ gives

$$U(\vec{r}, t) \simeq \frac{k_e q Q(t)}{r}.$$

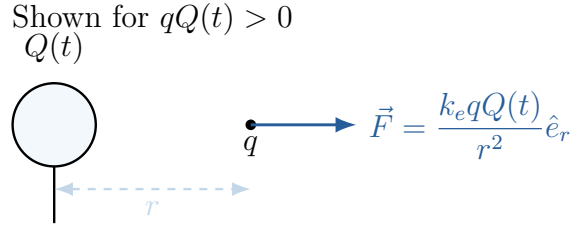


Fig. 9: Quasistatic Coulomb model: the particle can move through the potential, and the controlled source charge can also change the potential at a fixed position.

There are two ways for the potential energy to change. Moving the particle changes r even if Q is fixed. Changing $Q(t)$ changes the potential energy even if the particle is held at a fixed position. Within this model, the total differential (全微分) is

$$dU = \underbrace{-\frac{k_e q Q(t)}{r^2} dr}_{\text{position changes}} + \underbrace{\frac{k_e q}{r} \dot{Q}(t) dt}_{\text{source changes}}.$$

Finite propagation speed. Electromagnetic changes propagate at the speed of light c in vacuum. Using the instantaneous $Q(t)$ above assumes that the source changes little during the light-travel time r/c ; magnetic and radiative effects are also neglected. A full electromagnetic treatment accounts for propagation, the source current and field energy.

Total and partial time derivatives describe different changes. The partial derivative $\partial U / \partial t$ holds position fixed and describes the second effect alone. The total derivative dU / dt follows the moving particle and includes both effects. For a general $U(x, y, z, t)$, the chain rule gives

$$\begin{aligned} \frac{dU}{dt} &= \frac{\partial U}{\partial x} \dot{x} + \frac{\partial U}{\partial y} \dot{y} + \frac{\partial U}{\partial z} \dot{z} + \frac{\partial U}{\partial t} \\ &= -\vec{F} \cdot \vec{v} + \frac{\partial U}{\partial t}. \end{aligned}$$

For the same force, the work–energy theorem gives

$$\begin{aligned} \frac{dE_K}{dt} &= \frac{d}{dt} \left(\frac{m}{2} v^2 \right) \\ &= m \dot{\vec{v}} \cdot \vec{v} = \vec{F} \cdot \vec{v}. \end{aligned}$$

Adding the two equations yields the general identity

$$\boxed{\frac{d}{dt}(E_K + U) = \frac{\partial U}{\partial t}.$$

For the quasistatic charge model, the explicit time-dependent term is

$$\frac{\partial U}{\partial t} = \frac{k_e q}{r} \dot{Q}(t).$$

The particle's mechanical energy can therefore change as the source changes. The controlled source and its external circuit must be included in the full energy bookkeeping.

9.5.2 Moving a spring anchor

The same distinction appears in a mechanical model. Let the prescribed anchor position be $a(t)$, with the natural spring length absorbed into the definition of $x - a$. For a horizontal mass and spring,

$$U(x, t) = \frac{k}{2}[x - a(t)]^2, \quad F_x = -k[x - a(t)].$$

The general identity gives

$$\frac{d}{dt}(E_K + U) = \frac{\partial U}{\partial t} = -k[x - a(t)]\dot{a}(t).$$

Within the ideal massless-spring model, moving the anchor changes the potential at fixed x and transfers energy through the system boundary.

Energy is the scalar invariant tied to time independence. For many-particle systems the next simplification is vectorial: internal forces cancel from the total momentum balance.