

Lecture 8: Conservation law one: energy conservation (part I)

Invariants behind change. After Newton gives equations of motion, the next question is what survives the motion. The old debate over vis viva becomes the modern energy viewpoint.

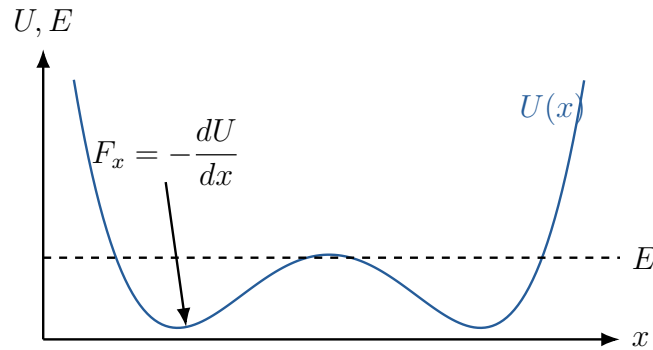


Fig. 1: Energy landscape: $F_x = -dU/dx$, and motion at energy E is allowed where $U(x) \leq E$.

Newton's law determines motion through second-order differential equations and initial conditions. Nevertheless, the perspective of conservation laws is a major development of Newtonian mechanics whose importance is at least two-fold. One side is empirical: The second-order differential equations arising from Newton's second law are often difficult to solve. If we can find a constant of motion, the second-order differential equations can often be reduced to first-order equations, whose solution would be much easier. The other side is more profound: **conservation laws reflect symmetries of the system's dynamics.** Time-translation, space-translation, and rotational invariance of its action lead to energy, momentum, and angular-momentum conservation, respectively. These statements refer to the chosen system and its interactions. A prescribed time-dependent drive can break its time-translation symmetry; a nonuniform external field can break its spatial translation symmetry.

8.1 Momentum and living force (vis viva)

Momentum and kinetic energy describe different aspects of motion. The historical *vis viva* (活力) was proportional to mv^2 .^[1]

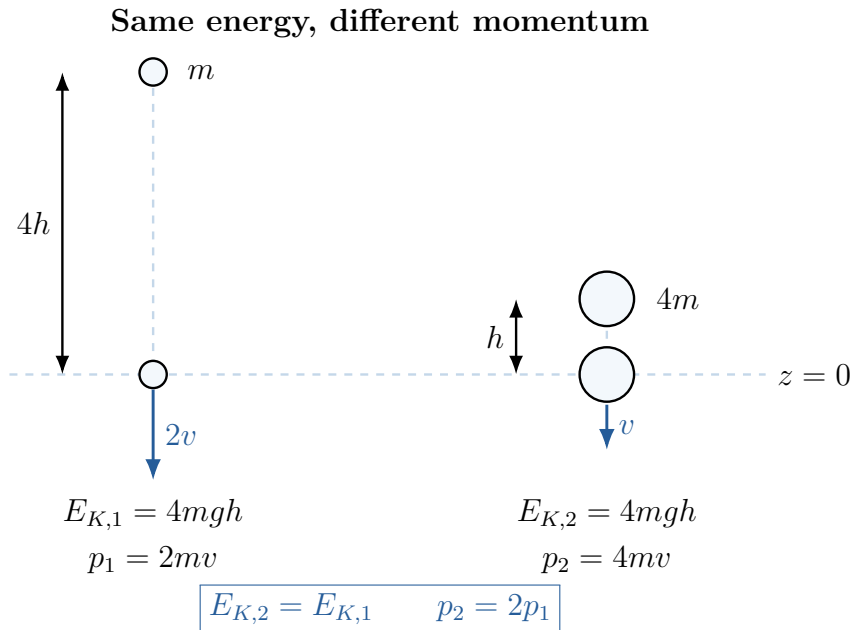


Fig. 2: Two masses released from rest, with negligible air resistance. At the reference level, $v = \sqrt{2gh}$; their kinetic energies are equal, while their momenta differ.

For constant mass in an inertial frame, **impulse and work give different changes**:

$$\Delta \vec{p} = \int_{t_i}^{t_f} \vec{F} dt, \quad \Delta E_K = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}.$$

With the **same constant net stopping force**, the two stopping distances are equal and $t_2 = 2t_1$. Equal momentum also need not imply equal energy: $E_K = p^2/(2m)$.

For a system of particles,

$$E_K = \frac{1}{2} \sum_i m_i v_i^2. \quad (8.1)$$

A missing particle: Pauli's neutrino. In beta decay (β 衰变), electrons emerge with a continuous range of kinetic energies. For a parent nucleus at rest and a fixed daughter state, a two-body decay would give a fixed electron energy. Where does the unobserved energy go?

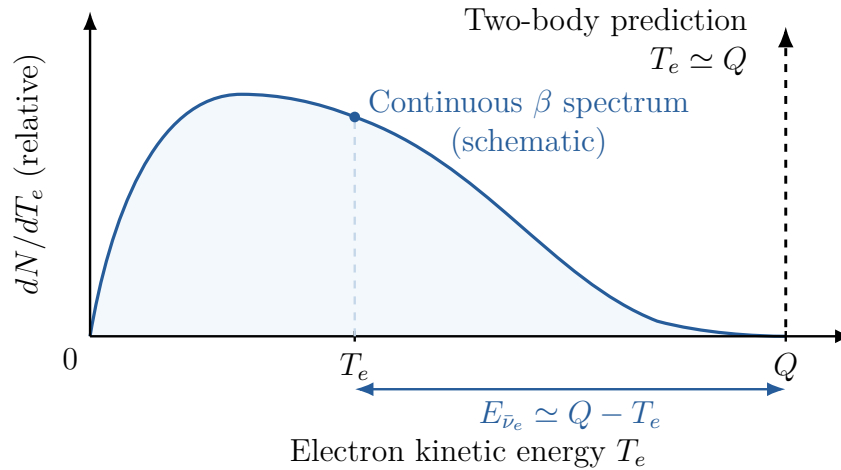


Fig. 3: The beta-spectrum puzzle for a single nuclear transition. The curve is schematic, not experimental data. Daughter recoil and neutrino-mass effects are neglected in the endpoint and energy-sharing labels.

On 4 December 1930, Wolfgang Pauli (泡利) proposed a light neutral particle that could escape unseen and share the released energy with the electron, preserving total energy and momentum.^[2] In modern notation, beta-minus decay is

$$X \longrightarrow Y + e^- + \bar{\nu}_e,$$

$$Q = T_e + T_Y + E_{\bar{\nu}_e}.$$

Here Q is the energy released after accounting for the daughter and electron rest energies; T_Y is the daughter's recoil kinetic energy. The electron's share varies while the total is conserved. In 1956, Cowan, Reines, and their colleagues detected reactor electron antineutrinos.^[3] **Conservation laws can point to an unseen particle; independent detection tests the hypothesis.**

8.2 Gravitational potential energy and reversible machines

How much can a falling mass lift? Compare machines with the same driving mass m , drop h , and lift h' . All loads start and finish at rest; ideal transfers supply no energy.

Postulate: no perpetual-motion machine exists.^[4] A cyclic machine cannot raise a weight without drawing on an energy source. Its configuration and stored energy must be restored after each cycle.

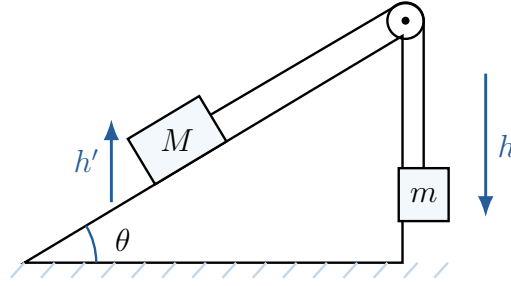


Fig. 4: A pulley and an incline: lowering m raises M . The vertical displacements obey $h' = h \sin \theta$.

$$\frac{h'}{h} = \sin \theta, \quad (8.2)$$

$$R = \frac{M}{m}. \quad (8.3)$$

Reversible: the frictionless, balanced machine runs either way quasistatically (准静态), with arbitrarily small starting and stopping pushes. **Irreversible:** friction dissipates energy as heat; returning the system requires extra work.

Why does a reversible machine give the maximum lift?

1. Suppose machine i lifts $M_i > M_r$, where M_r is the reversible machine's load.
2. Run i forward. Leave $\Delta M = M_i - M_r$ raised; transfer the remaining loads at fixed height and run the reversible machine backward.
3. The driving mass m and load M_r return, while ΔM stays raised — a repeatable lift with no energy supply, excluded by the postulate.

Thus $R_i \leq R_r$. For two reversible machines, either can run backward, so **their ratios are equal, regardless of design.**

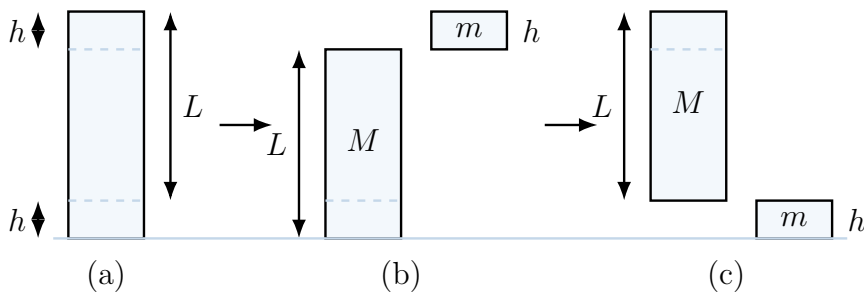


Fig. 5: Horizontal rearrangement of a uniform column. Between configurations (b) and (c), the small piece descends by L while the large piece rises by h .

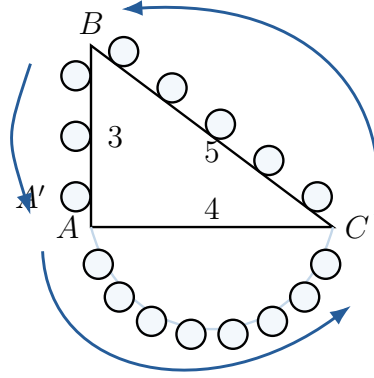
Finding the ratio. A uniform column has height $L + h$. Configurations (b) and (c) each come from (a) by horizontal rearrangement at unchanged heights. Comparing them, m descends by L and M rises by h . Uniform density gives

$$\frac{m}{M} = \frac{h}{L} \implies mgL - Mgh = 0. \quad (8.4)$$

Gravitational potential energy, in a uniform field:

$$E_p = \sum_i m_i g z_i. \quad (8.5)$$

Here z_i is height above $z = 0$. A balanced reversible exchange has $\Delta E_p = 0$; gravity does work $W_g = -\Delta E_p$. For the incline, $Mgh' = mgh$ gives $R = h/h' = 1/\sin \theta$.



The lower segment balances itself.

Fig. 6: Stevin's chain on a 3 : 4 : 5 triangle. The beads mark equal-mass segments of an ideal uniform chain: three vertical segments balance five on the incline.

Why must the chain balance? Treat the chain as uniform and continuous, on a frictionless support. Sliding it leaves its mass distribution unchanged. A persistent drive could then raise a load cycle after cycle, with the chain's configuration and speed restored and no energy source. The perpetual-motion postulate excludes such a drive. The hanging part, whose ends are at equal heights, contributes no net driving work. Thus the five incline segments balance the three vertical segments.^[4]

Let m_b be the mass of one marked segment and $F_{\parallel}$ its downhill gravitational force on the incline. An imagined chain displacement δs gives

$$\begin{aligned} \delta W &= (3m_b g - 5F_{\parallel}) \delta s = 0, \\ F_{\parallel} &= \frac{3}{5} m_b g = m_b g \sin \theta. \end{aligned}$$

Conservation plus geometry predicts the force along an incline. This connects to *virtual work* (虚功): equilibrium requires zero net work for an allowed infinitesimal displacement. One principle links inclines, pulleys, and levers. The assumptions and predictions still need experimental checks.

8.3 Kinetic energy

We only considered infinitesimally slow motion above, hence, only the gravitational potential energy is needed. Nevertheless, physical intuition tells us that an object falling from a height will increase its speed. Now we will use the postulate of the impossibility of perpetual machine to study the speed of an object after falling a height.

Consider the frictionless tracks in Fig. 7, treating the ball as a sliding particle with no rotational energy. Each path has the same vertical drop. Again we divide these kinetic-energy machines into reversible and irreversible classes.

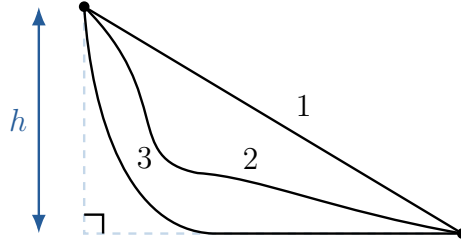


Fig. 7: Frictionless tracks with the same vertical drop give the same final speed for a particle released from rest.

Reversible kinetic-energy machines mean that if the motion of the ball at the bottom is reversed but with the same speed, it will reach where it drops. Again following similar reasonings in the previous section, we conclude that the reversible machines exhibit the largest speeds, and **all reversible machines exhibit the same speed**. Otherwise, we could lift the ball without paying any effort.

Then we can use the straight-line slope to calculate the speed after it moves to the bottom. The force F to pull the ball with the mass M on the slope can be balanced by a mass $m = M \sin \theta$, hence,

$$a = \frac{F}{M} = \frac{mg}{M} = g \sin \theta. \quad (8.6)$$

Then

$$v^2 = 2al = 2gl \sin \theta = 2gh, \quad (8.7)$$

which is independent of the slope angle. We arrive at

$$\frac{1}{2}mv^2 = mgh. \quad (8.8)$$

The right-hand side is the gravitational potential-energy drop, hence, if we interpret the left-hand side as the gain of another type of energy — kinetic energy, then the total energy should be conserved. Then we define the kinetic energy as

$$E_K = \frac{1}{2}mv^2. \quad (8.9)$$

During this process, we have the **conservation of mechanical energy** in a reversible system which is defined as

$$E_M = E_K + E_p = \frac{1}{2}mv^2 + mgz \quad (8.10)$$

8.4 Work and energy

For a particle of constant mass in an inertial frame, take the scalar product of Newton's law with the displacement:

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= m \frac{d\vec{v}}{dt} \cdot \vec{v} dt \\ &= m(v_x dv_x + v_y dv_y + v_z dv_z) \\ &= d\left[\frac{m}{2}(v_x^2 + v_y^2 + v_z^2)\right].\end{aligned}$$

Thus the **work–energy theorem** is

$$\boxed{dE_K = \vec{F} \cdot d\vec{r}}, \quad E_K = \frac{1}{2}mv^2.$$

8.4.1 Line integrals

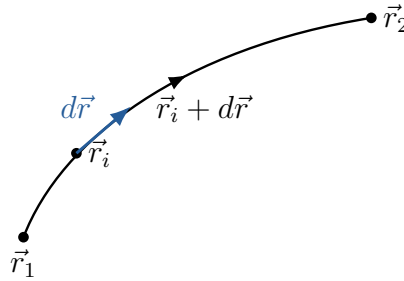


Fig. 8: The infinitesimal displacement $d\vec{r}$ is tangent to the path from $\vec{r}_1$ to $\vec{r}_2$.

$$\begin{aligned}E_{K_2} - E_{K_1} &= \lim_{\max|\Delta\vec{r}_i| \rightarrow 0} \sum_i \vec{F}_i \cdot \Delta\vec{r}_i \\ &= \int_1^2 d\vec{r} \cdot \vec{F}\end{aligned}$$

Along the path from $1 \rightarrow 2$.

Examples.

(1) Projectile motion

$$E_{K_2} - E_{K_1} = \int_1^2 d\vec{r} \cdot \vec{G}$$

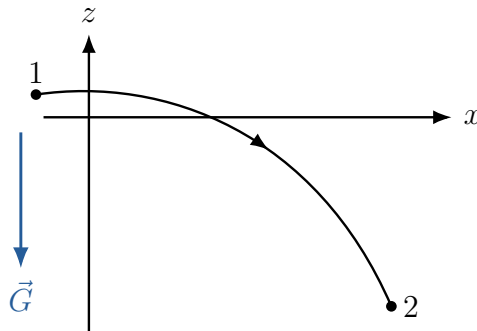


Fig. 9: Work by gravity depends on the vertical drop, not on the projectile's curved path.

Since

$$\begin{aligned}\vec{G} &= -mg\hat{z} \\ \Rightarrow \Delta E_K &= \int_{z_1}^{z_2} dz(-mg) = -mg(z_2 - z_1) \\ &\Rightarrow \Delta E_K + \Delta(mgz) = 0, \text{ i.e. } \Delta(E_K + mgz) = 0.\end{aligned}$$

(2) Uniform circular motion

$$\begin{aligned}E_{K_2} - E_{K_1} &= \int_1^2 d\vec{r} \cdot \vec{F} = 0 \\ \Rightarrow \Delta E_K &= 0\end{aligned}$$

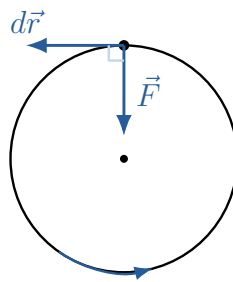


Fig. 10: A radial force is perpendicular to the displacement in uniform circular motion and does no work.

(3) Elastic restoring force

$$\begin{aligned}F &= -kx \\ \Delta E_K &= \int_{x_1}^{x_2} F(x) dx = -\frac{k}{2}(x_2^2 - x_1^2) \\ &\Rightarrow \Delta \left(\frac{1}{2}mv^2 + \frac{k}{2}x^2 \right) = 0 \\ \Rightarrow \Delta(\text{kinetic energy} + \text{elastic energy}) &= 0\end{aligned}$$

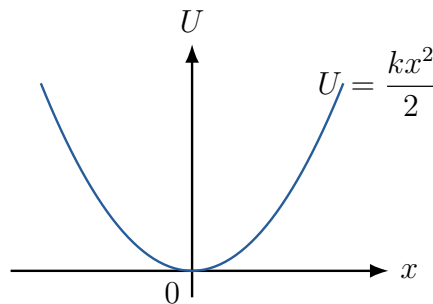


Fig. 11: Elastic potential energy $U(x) = kx^2/2$.

8.4.2 Conservative forces

Consider a continuous force field $\vec{F}(\vec{r})$ that depends only on position. It is **conservative** if its work between any two points is independent of the path. This property allows us to assign a potential energy to each position. First, we must check path independence.

Even a radial force can have path-dependent work. For example, let

$$\vec{F} = f(r, \theta) \hat{e}_r$$

For path 1, the displacement is tangential at fixed radius R_1 :

$$W_1 = \int_{\theta_{13}}^{\theta_{24}} f(R_1, \theta) \hat{e}_r \cdot (R_1 \hat{e}_\theta d\theta) = 0.$$

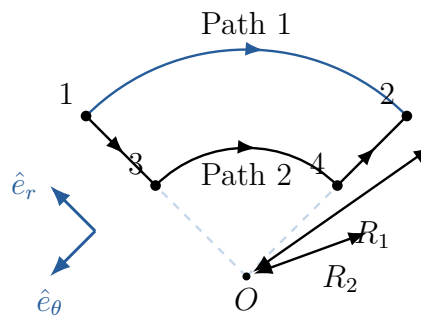


Fig. 12: Two routes between points 1 and 2: an outer arc and the route $1 \rightarrow 3 \rightarrow 4 \rightarrow 2$.

For path 2, the inner arc also contributes zero work, while the two radial segments give

$$\begin{aligned} W_2 &= \int_{R_1}^{R_2} f(r, \theta_{13}) dr + \int_{R_2}^{R_1} f(r, \theta_{24}) dr \\ &= \int_{R_2}^{R_1} [f(r, \theta_{24}) - f(r, \theta_{13})] dr. \end{aligned}$$

A radial direction alone does not ensure a conservative force: angular dependence of its magnitude can make $W_2 \neq W_1$.

Path independence is equivalent to zero work around every closed loop:

$$\oint \vec{F} \cdot d\vec{r} = 0$$

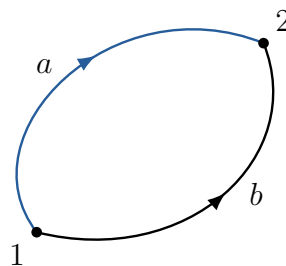


Fig. 13: Two paths a and b from 1 to 2. Going forward along a and backward along b forms a closed loop.

Proof. Let W_a and W_b be the work along two paths from 1 to 2. Traversing a forward and b backward gives

$$\oint_{a-b} \vec{F} \cdot d\vec{r} = \int_a \vec{F} \cdot d\vec{r} - \int_b \vec{F} \cdot d\vec{r} = W_a - W_b.$$

If the work is path independent, $W_a = W_b$ and every loop integral vanishes. Conversely, if every closed-loop integral vanishes, then $W_a = W_b$ for every pair of paths with the same endpoints.

Defining potential energy. Choose a fixed reference point $\vec{r}_0$ and specify $U(\vec{r}_0)$. For a conservative force, define

$$U(\vec{r}) = U(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{\ell}.$$

Path independence makes U a single-valued function of position: every route to the same endpoint gives the same value. Choosing a different reference value adds a constant to U ; potential-energy differences stay the same.

To relate the energies at two points A and B , choose a route from $\vec{r}_0$ through A to B . The common part of the two integrals cancels:

$$\begin{aligned} U(B) - U(A) &= - \left(\int_{\vec{r}_0}^A \vec{F} \cdot d\vec{\ell} + \int_A^B \vec{F} \cdot d\vec{\ell} \right) + \int_{\vec{r}_0}^A \vec{F} \cdot d\vec{\ell} \\ &= - \int_A^B \vec{F} \cdot d\vec{\ell}. \end{aligned}$$

The minus sign means that **positive work by the conservative force reduces potential energy**.

From work to a local relation. Let B approach A along a straight segment, with $\Delta\vec{r} = \vec{r}_B - \vec{r}_A$. Continuity of the force gives, to first order,

$$\Delta U = - \int_A^B \vec{F} \cdot d\vec{\ell} \simeq -\vec{F}(A) \cdot \Delta\vec{r}.$$

This first-order change is linear in the displacement, so U is differentiable, with

$$dU = -\vec{F} \cdot d\vec{r} = -F_x dx - F_y dy - F_z dz.$$

Why the gradient appears. For the position function $U(x, y, z)$, the total differential (全微分) is

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz.$$

Both expressions for dU hold for **every direction of displacement**. Moving only along x , with $dy = dz = 0$, gives $F_x = -\partial U / \partial x$. The other two directions give

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}.$$

The *gradient* (梯度) collects these spatial derivatives into a vector:

$$\begin{aligned} \vec{\nabla} U &= \frac{\partial U}{\partial x} \hat{e}_x + \frac{\partial U}{\partial y} \hat{e}_y + \frac{\partial U}{\partial z} \hat{e}_z, \\ \vec{F} &= -\vec{\nabla} U. \end{aligned}$$

In one dimension, this reduces to $F_x = -dU/dx$.

Mechanical-energy conservation follows. If this conservative force is the net force, the work–kinetic-energy theorem gives

$$\Delta E_K = \int_A^B \vec{F} \cdot d\vec{\ell} = -\Delta U, \quad \Delta(E_K + U) = 0.$$

Thus path independence lets us define U , its spatial derivatives give the force, and the work–kinetic-energy theorem gives a conserved mechanical energy.

8.4.3 Coulomb and gravitational potentials

In SI units, write $k_e = 1/(4\pi\epsilon_0)$. A fixed point charge Q exerts

$$\vec{F}(\vec{r}) = \frac{k_e q Q}{r^2} \hat{e}_r.$$

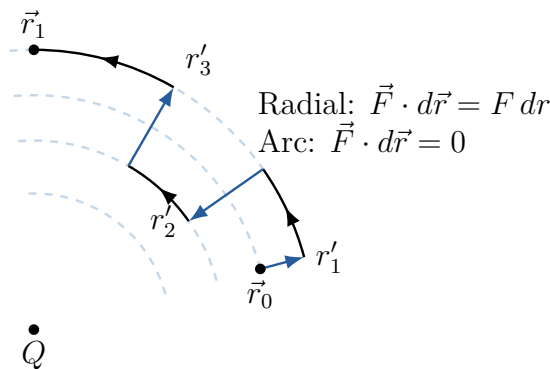


Fig. 14: A path resolved into radial segments and circular arcs. Only radial segments contribute to the work of this central force.

The radial integrals telescope, regardless of the intermediate radii:

$$\begin{aligned} U(r_1) - U(r_0) &= - \int_{r_0}^{r_1} F(r) dr \\ &= -k_e q Q \int_{r_0}^{r_1} \frac{dr}{r^2} \\ &= k_e q Q \left(\frac{1}{r_1} - \frac{1}{r_0} \right). \end{aligned}$$

Choosing $U(\infty) = 0$ gives the potential energy and electric potential:

$$U(r) = \frac{k_e q Q}{r}, \quad V(r) = \frac{U(r)}{q} = \frac{k_e Q}{r}.$$

Similarly, outside a spherical mass M , the gravitational potential energy of a mass m is

$$U(r) = -\frac{GMm}{r}$$

Then the conservation of energy is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

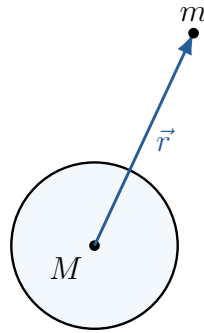


Fig. 15: The distance r is measured from the center of a spherical gravitating body; the exterior potential is $-GMm/r$.

References

- [1] G. W. Leibniz, *Specimen dynamicum* (1695), Part I. J. Bennett's modernized translation, *A specimen of dynamics*. <https://www.earlymoderntexts.com/assets/pdfs/leibniz1695b.pdf>
- [2] W. Pauli, letter to the participants of the Tübingen radioactivity meeting, 4 December 1930. French translation reproduced by LAPP, CNRS/IN2P3. <https://lappweb.in2p3.fr/neutrinos/plettre.html>
- [3] C. L. Cowan Jr., F. Reines, F. B. Harrison, H. W. Kruse, and A. D. McGuire, *Detection of the free neutrino: a confirmation*, *Science* **124**, 103–104 (1956). <https://doi.org/10.1126/science.124.3212.103>
- [4] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. I, Ch. 4, Secs. 4–1, 4–2, and 4–3. The reversible-machine reasoning is a teaching argument. Stevin's chain and virtual work: Sec. 4–2, Fig. 4–4 and the weighted-rod example. https://www.feynmanlectures.caltech.edu/I_04.html