

Lecture 7: Galilean transformation, inertial and non-inertial frames

From orbits to the observer. In L6, gravity determined the orbit. An observer carried along by Earth is also accelerating under gravity. We now transform Newton's laws between observers, then use the difference in lunar gravity across Earth to understand tides.

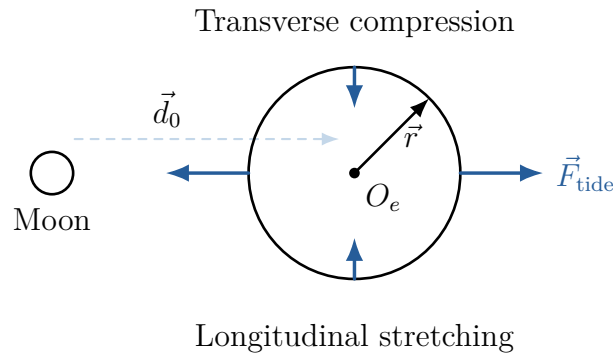


Fig. 1: Differential lunar gravity stretches Earth along the Earth–Moon line.

Outline

(1) Inertial frames, Galilean relativity

The laws of Newtonian mechanics have the same form in every inertial frame. Internal mechanical experiments cannot determine a uniform velocity relative to an external frame.

(2) Non-inertial frames — modification of Newton's law

— tidal effect

— Approximately twice-daily tides — a quadrupolar response

— solar tide and lunar tide

Constructive superposition — spring tide; destructive — neap tide.

Tidal locking.

7.1 Inertial frames

An inertial frame is one in which a force-free particle has constant velocity and Newton's second law holds using interaction forces alone. Newton's 1st law essentially states the existence of inertial frames. In practice, an ideal perfectly inertial frame does not exist. The Earth is a pretty good concrete approximation, but it spins and orbits the Sun; the Sun also moves in the Milky Way.

(1) Suppose reference frame A is inertial. Another frame B is static with respect to A but shifted spatially. Take parallel axes with the same units and orientation. The transformation between position vectors is

$$\begin{cases} \vec{r}_A(t) = \vec{r}_{AB} + \vec{r}_B(t'), \\ t = t'. \end{cases}$$

In this case

$$\begin{aligned}\frac{d}{dt}\vec{r}_A(t) &= \frac{d}{dt'}\vec{r}_B(t'), \\ \frac{d^2}{dt^2}\vec{r}_A(t) &= \frac{d^2}{dt'^2}\vec{r}_B(t'), \\ \vec{F}_A &= \vec{F}_B.\end{aligned}$$

If in the A -frame $\vec{F}_A = m\vec{a}_A$, then $\vec{F}_B = m\vec{a}_B$ in the B -frame as well. Therefore B is also **inertial**.

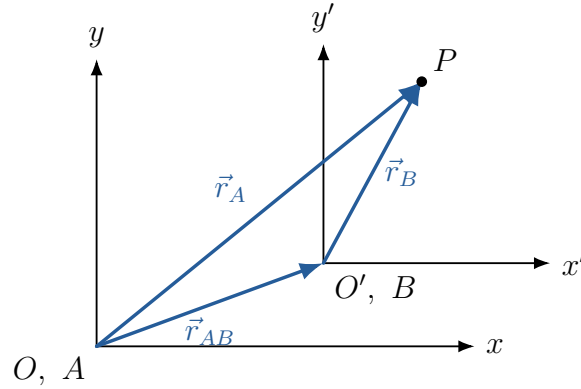


Fig. 2: Position vectors in two spatially displaced inertial frames.

(2) Suppose the frame B has the same origin and a fixed orientation rotated by angle θ with respect to A . The coordinate transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad t = t'.$$

Hence

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = R_z(\theta) \begin{pmatrix} v_{x'} \\ v_{y'} \end{pmatrix}, \quad \begin{pmatrix} a_x \\ a_y \end{pmatrix} = R_z(\theta) \begin{pmatrix} a_{x'} \\ a_{y'} \end{pmatrix},$$

and force, being a vector, transforms identically:

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = R_z(\theta) \begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix}, \quad R_z(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Thus if $\begin{pmatrix} F_x \\ F_y \end{pmatrix} = m \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ holds in A , then $\begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix} = m \begin{pmatrix} a_{x'} \\ a_{y'} \end{pmatrix}$ holds in B . So B is inertial.

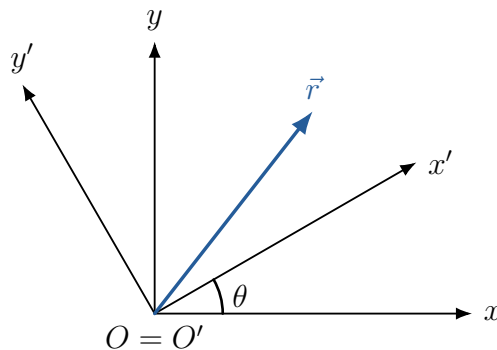


Fig. 3: A fixed rotation changes vector components while preserving the vector.

(3) If B moves with constant velocity $\vec{v}_0 = v_0 \hat{x}$ relative to A , with origins coincident at $t = 0$, they are related by the Galilean boost:

$$\begin{cases} x = x' + v_0 t \\ y = y' \\ t = t' \end{cases} \Rightarrow \begin{cases} v_x = v'_x + v_0 \\ v_y = v'_y \end{cases} \Rightarrow \begin{cases} a_x = a'_x \\ a_y = a'_y \end{cases}$$

Therefore, if $\vec{F} = m\vec{a}$ is valid in A , it remains valid in B under Galilean transformations. So far we have shown that **translations, fixed rotations, and Galilean boosts** preserve Newton's second law. In fact, the collection of these operations forms a symmetry group — the **Galilean group**, including time translations.

Galilean relativity: Mechanical laws are the same in all inertial reference frames. No inertial frame is special. You may compare it with the **Lorentz boost** from **special relativity** in the future.

7.2 Non-inertial frames

If a reference frame is rotating or accelerating with respect to an inertial frame, then Newton's laws are no longer valid in their simple form. It's then called a **non-inertial frame**. An accelerating car, train, or elevator is non-inertial; strictly speaking, Earth is also non-inertial due to its spin and orbital motion. It is convenient to use non-inertial frames because measurements are performed in them; then we must modify Newton's second law by adding **fictitious (inertial) forces** (惯性力) .

Feynman perspective: Pseudo-forces and the observer.

Inertial forces are not new interactions between bodies. They are the price of describing motion in a frame whose axes accelerate. The practical rule is simple: write Newton's law in an inertial frame first, transform coordinates to the frame used by the observer, and move the extra acceleration terms to the force side. In a purely translating frame this gives $\vec{F}_{\text{inertial}} = -m\vec{A}$; in a rotating frame it also produces centrifugal and Coriolis terms. These useful forces are frame-dependent.^[1]

We first develop translating frames with nonrotating axes. L14 introduces the derivative rule for rotating axes and applies it to rigid bodies.

7.2.1 Acceleration without rotation

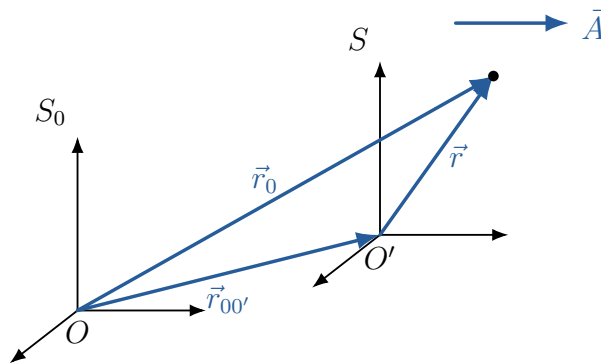


Fig. 4: Parallel axes in a translating frame whose acceleration is $\vec{A}$.

S_0, S Inertial and translating non-inertial frames,

$\vec{r}_0, \vec{r}$ Position vectors in S_0 and S ,

$\vec{r}_{00'}$ Vector from the origin of S_0 to the origin of S .

Newton's law in S_0 :

$$m\ddot{\vec{r}}_0 = \vec{F}$$

Relation between frames:

$$\vec{r}_0 = \vec{r}_{00'} + \vec{r} \Rightarrow \ddot{\vec{r}}_0 = \ddot{\vec{r}}_{00'} + \ddot{\vec{r}} = \vec{A} + \ddot{\vec{r}}.$$

Law in frame S :

$$m\ddot{\vec{r}} = \vec{F} - m\vec{A} = \vec{F} + \vec{F}_{\text{inertial}}, \quad \vec{F}_{\text{inertial}} = -m\vec{A}.$$

7.2.2 Pendulum in an accelerating car

For constant horizontal acceleration A , the equilibrium and the frequency of small oscillations about it follow from effective gravity.

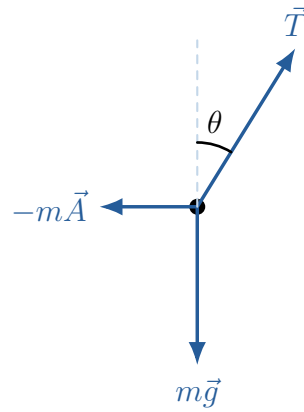


Fig. 5: Forces on the bob in the accelerating car frame.

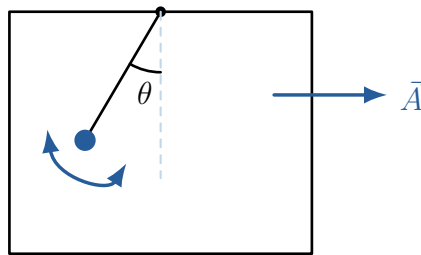


Fig. 6: The equilibrium string tilts opposite to the acceleration of the car.

$$m\ddot{\vec{r}} = \vec{T} + \vec{G} + \vec{F}_{\text{in}} = \vec{T} + m(\vec{g} - \vec{A}), \quad \vec{g}_{\text{eff}} = \vec{g} - \vec{A},$$

$$|\vec{g}_{\text{eff}}| = \sqrt{g^2 + A^2}, \quad \tan \theta = \frac{A}{g},$$

$$\omega = \sqrt{\frac{|\vec{g}_{\text{eff}}|}{l}} = \omega_0 \left(1 + \frac{A^2}{g^2}\right)^{1/4}, \quad \omega_0 = \sqrt{\frac{g}{l}}.$$

7.3 Application: tides due to lunar gravity

A famous Tang dynasty poem by Zhang Ruoxu (张若虚), *Spring River, Flower, Moon, Night* (《春江花月夜》), beautifully captures the connection between tides and the rising moon:

原文	English translation
春江潮水连海平，海上明月共潮生。	The spring river rises with the sea tide, and over the sea, the bright moon ascends together with the tide.
滟滟随波千万里，何处春江无月明？	Its shimmering light follows the waves for thousands of miles — where on the river does the moon not shine bright?

This verse illustrates the poetic perception of the *moon–tide connection*, long before the physics of tidal forces was fully understood.

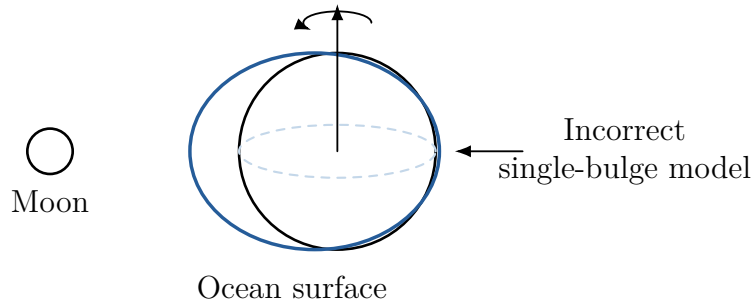


Fig. 7: A single bulge is an incorrect model: it omits the acceleration of Earth.

Treating Earth as fixed while applying the Moon's pull only to the ocean leads to this incorrect single-bulge picture. Earth and the ocean both accelerate toward the Moon; their relative acceleration determines the tide.

Use a translating, nonrotating frame centered on Earth; this isolates the lunar tide without Earth-spin inertial terms. Let $\vec{d}_0$ point from the Moon to Earth's center, $\vec{r}$ from that center to a test mass, and $\vec{d} = \vec{d}_0 + \vec{r}$ from the Moon to the test mass. The Moon–Earth system orbits its center of mass. The Earth's lunar acceleration is

$$\vec{A}_{\text{earth}} = -\frac{G M_{\text{moon}}}{d_0^2} \hat{d}_0.$$

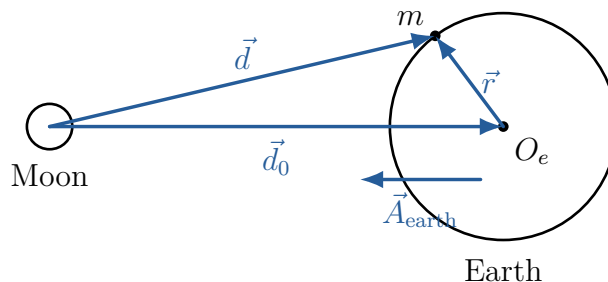


Fig. 8: The vectors obey $\vec{d} = \vec{d}_0 + \vec{r}$; Earth accelerates toward the Moon.

$$\begin{cases} m\ddot{\vec{r}} = \vec{F} - m\vec{A}_{\text{earth}}, \\ \vec{F} = m\vec{g} + \vec{F}_{\text{other}} - \frac{GM_{\text{moon}}m}{d^2} \hat{d}. \end{cases}$$

Thus

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{\text{other}} + \vec{F}_{\text{tide}},$$

$$\vec{F}_{\text{tide}} = -GM_{\text{moon}}m \left[\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2} \right].$$

Let $\vec{d} = \vec{d}_0 + \vec{r}$, with $|\vec{r}|/d_0 \sim 6.4 \times 10^3 \text{ km} / 3.8 \times 10^5 \text{ km} \approx 0.02 \ll 1$. Then

$$\begin{aligned} \frac{\hat{d}}{d^2} &= \frac{\vec{d}_0 + \vec{r}}{d_0^3} \left(1 + \frac{2\vec{d}_0 \cdot \vec{r}}{d_0^2} + \frac{r^2}{d_0^2} \right)^{-3/2} \\ &\simeq \frac{\vec{d}_0 + \vec{r}}{d_0^3} \left(1 - \frac{3\vec{d}_0 \cdot \vec{r}}{d_0^2} \right) \\ &\simeq \frac{\vec{d}_0}{d_0^3} + \frac{\vec{r}}{d_0^3} - \frac{3\vec{d}_0(\vec{d}_0 \cdot \vec{r})}{d_0^5}. \end{aligned}$$

so

$$\vec{F}_{\text{tide}} \simeq \frac{GM_{\text{moon}}m}{d_0^2} \left[-\frac{\vec{r}}{d_0} + \frac{3\hat{d}_0(\hat{d}_0 \cdot \vec{r})}{d_0} \right].$$

Interpretation: the tidal acceleration is the difference between lunar gravity at $\vec{r}$ and at Earth's center O_e . To first order it is the gravity gradient (引力梯度) acting on the separation:

$$\delta\vec{g}_{\text{moon}} = \vec{g}_{\text{moon}}(\vec{d}_0 + \vec{r}) - \vec{g}_{\text{moon}}(\vec{d}_0) \simeq (\vec{r} \cdot \nabla) \vec{g}_{\text{moon}}|_{\vec{d}_0}.$$

In a uniform gravity field, the relative gravitational acceleration vanishes:

$$\vec{F} = m\vec{g} - m\vec{A} = m(\vec{g} - \vec{g}) = 0.$$

Falling together, separating slightly. Release two nearby test masses inside a freely falling laboratory. Their common gravitational acceleration is removed by following the laboratory. A spatial variation of gravity still changes their separation. For the lunar contribution calculated above, the separation is stretched along the Earth–Moon line and compressed transversely. A local free-fall frame removes the uniform part of gravity; the relative motion reveals the tidal part.^[2]

Direction of the tidal force (quadrupolar response):

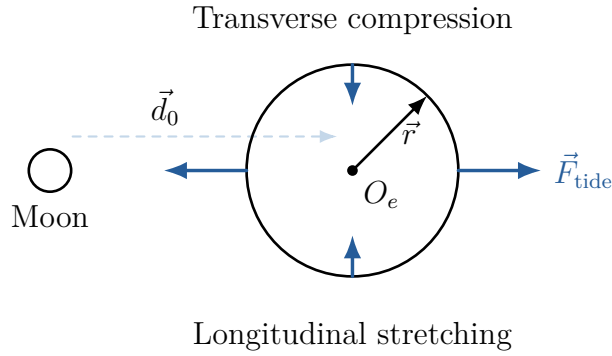


Fig. 9: The tidal field stretches along the lunar direction and compresses transversely.

$$\vec{F}_{\text{tide}} \simeq \frac{GM_{\text{moon}}m}{d_0^2} \underbrace{[-\hat{r} + 3\hat{d}_0(\hat{d}_0 \cdot \hat{r})]}_{\text{angular form factor}} \frac{|r|}{d_0}$$

For points on Earth, within the linear tidal approximation:

$$\begin{aligned} \hat{r} = \pm \hat{d}_0 : \quad \vec{F}_{\text{tide}} &\simeq \frac{GM_{\text{moon}}m|r|}{d_0^3} (2\hat{r}), \\ \hat{r} \perp \hat{d}_0 : \quad \vec{F}_{\text{tide}} &\simeq \frac{GM_{\text{moon}}m|r|}{d_0^3} (-\hat{r}). \end{aligned}$$

7.3.1 Solar contribution to tides

The Sun also produces tides. Although $M_{\odot} \gg M_{\text{moon}}$, the tidal effect scales as $1/r^3$.

Solar mass $M_{\odot} = 2 \times 10^{30}$ kg; lunar mass $M_{\text{moon}} = 7.3 \times 10^{22}$ kg.

Earth–Sun distance $d_{\text{ES}} = 1.5 \times 10^8$ km; Moon–Earth distance $d_{\text{EM}} = 3.8 \times 10^5$ km.

$$M_{\odot}/M_{\text{moon}} \approx 2.7 \times 10^7, \quad d_{\text{ES}}/d_{\text{EM}} \approx 3.9 \times 10^2 \Rightarrow (d_{\text{ES}}/d_{\text{EM}})^3 \approx 6 \times 10^7.$$

Hence,

$$\frac{M_{\odot}/d_{\text{ES}}^3}{M_{\text{moon}}/d_{\text{EM}}^3} \simeq \frac{2.7}{6} \approx 45\%.$$

So the solar tide is about half the lunar tide.

7.3.2 Periodicity of tides

The two-bulge equilibrium pattern gives a semidiurnal (半日) response: successive lunar high tides are about 12 hours 25 minutes apart. Coastlines and ocean dynamics also produce diurnal and mixed tides.^[3]

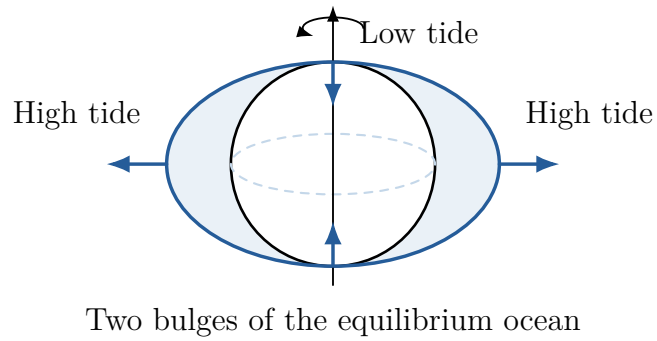


Fig. 10: Earth rotates beneath two idealized equilibrium tidal bulges.

For a spherical, rigid Earth covered by a shallow equilibrium ocean, the following approximate ranges neglect continents, ocean dynamics, and lunar declination. The estimated differences between high and low water are $\Delta h_{\text{moon}} \approx 54$ cm, and the Sun's effect is about half, $\Delta h_{\text{sun}} \approx 25$ cm. The varying lunar and solar contributions reinforce at spring tide (大潮) and partially cancel at neap tide (小潮). These estimates describe the idealized equatorial range, not a prediction for a particular coast.

7.3.3 Solar tide vs. lunar tide

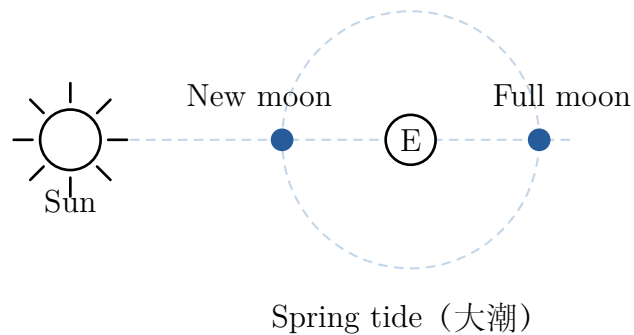


Fig. 11: At new and full moon, the solar and lunar tides reinforce.

$$\Delta h = \Delta h_{\text{moon}} + \Delta h_{\text{sun}} = 79 \text{ cm}$$

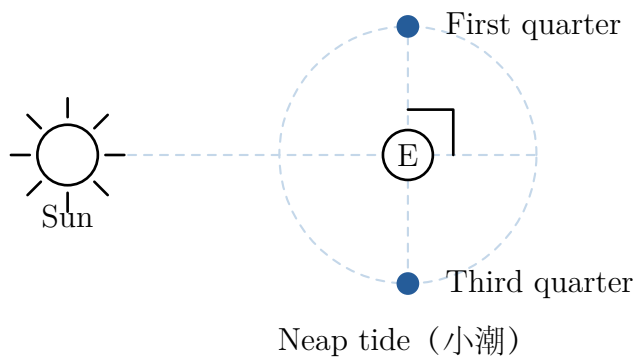


Fig. 12: At quarter moon, the solar and lunar tidal effects partially cancel.

$$\Delta h = \Delta h_{\text{moon}} - \Delta h_{\text{sun}} = 29 \text{ cm.}$$

7.4 Tidal locking

Tidal forces not only produce periodic ocean tides, but also gradually modify the long-term dynamics of the Earth–Moon system. The key effect is that the tidal bulges raised on Earth are dragged slightly ahead of the Earth–Moon line because Earth rotates faster than the Moon orbits. This misalignment creates a tangential gravitational component, which transfers angular momentum from the Earth's spin to the Moon's orbital motion.^[4]

7.4.1 Energy minimization argument

Let the angular velocities be:

ω_e : Earth's spin angular velocity,

ω_m : Moon's spin angular velocity,

ω_0 : Moon's orbital angular velocity.

Denote I_e, I_m as the rotational inertias of Earth and Moon, and $I_0 = mR^2$ as the orbital inertia of the Moon (with m the Moon's mass and R the Earth–Moon distance).

For aligned spin and orbital axes, the total angular momentum is

$$J = I_e\omega_e + I_m\omega_m + I_0\omega_0,$$

while the total kinetic energy is

$$E_K = \frac{1}{2} (I_e\omega_e^2 + I_m\omega_m^2 + I_0\omega_0^2).$$

By the Cauchy inequality,

$$J^2 \leq (I_e + I_m + I_0) (I_e\omega_e^2 + I_m\omega_m^2 + I_0\omega_0^2),$$

so that

$$E_K \geq \frac{J^2}{2(I_e + I_m + I_0)}.$$

Equality holds only when

$$\omega_e = \omega_m = \omega_0,$$

i.e. when Earth's spin, Moon's spin, and the Moon's orbital motion are synchronized. This is the **tidal locking configuration**, corresponding to the minimum kinetic energy at fixed total angular momentum *and fixed inertias*. This simplified argument illustrates synchronization. Orbital evolution also changes R and I_0 , and requires gravitational potential energy.

7.4.2 Current status of the Earth–Moon system

- The Moon is already tidally locked: $\omega_m = \omega_0$, so it always presents the same face to Earth.
- Earth is not yet locked. Its day (24 h) is much shorter than the Moon's sidereal orbital period (27.3 days), so $\omega_e \gg \omega_0$. The phase cycle (synodic month, 朔望月) is about 29.5 days.^[5]

- Tidal friction dissipates mechanical energy into heat, while approximately conserving the angular momentum of the Earth–Moon system. The Earth's rotation slows, and the Moon's orbital angular momentum increases, so the Moon slowly recedes ($\dot{R} \approx +3.8 \text{ cm/yr}$).
- Devonian coral growth bands have been interpreted as roughly 400 daily increments per year. If the year's duration in seconds was nearly unchanged, this corresponds to a day of about $365.25 \times 24/400 \simeq 22 \text{ h}$. The inference combines a biological growth clock with an orbital clock and is consistent with gradual spin-down.^[6]

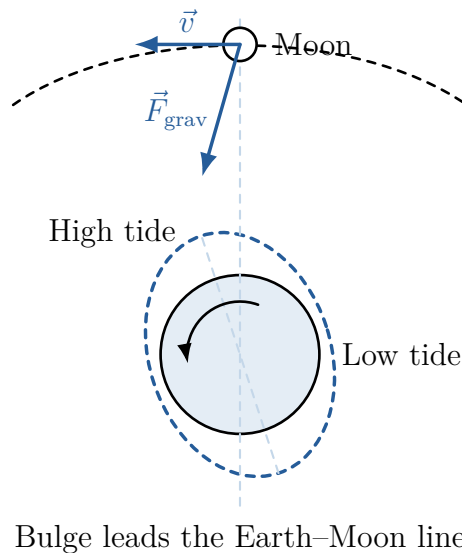


Fig. 13: A leading tidal bulge gives the Moon a forward gravitational force component.

7.4.3 Future evolution

An isolated, dissipative two-body model can approach mutual synchronization, $\omega_e = \omega_m = \omega_0$, when a stable synchronous state is accessible. In that configuration the same hemispheres would face each other. The real Earth–Moon system also feels solar torques, and stellar evolution limits extrapolation to such a distant future.

References

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