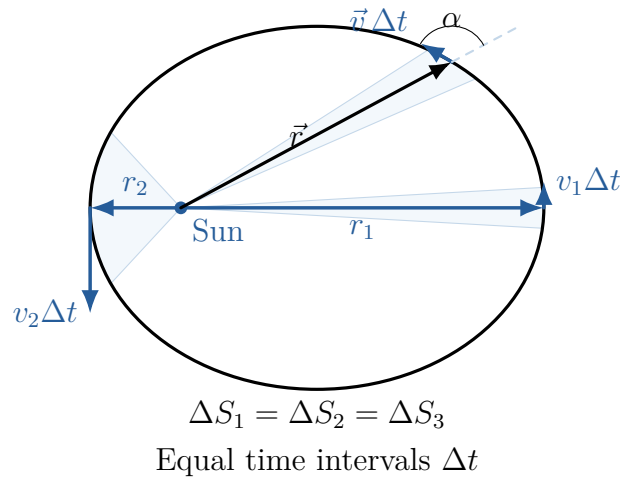


## Lecture 6: Kepler's problem

**From oscillations to orbits.** In L5, a linear restoring force made oscillations solvable. For planetary motion, a central force preserves angular momentum, and geometry becomes our main tool. We now connect Kepler's observational laws to Newton's inverse-square gravity.



**Fig. 1:** Equal areas are swept in equal time intervals.

Newton's account of planetary motion builds on Galileo's study of motion and Johannes Kepler's (约翰内斯·开普勒, 1571–1630) orbital laws. The solution of Kepler's problem connected terrestrial falling bodies and celestial orbits through universal gravitation.

### 6.1 Kepler's story

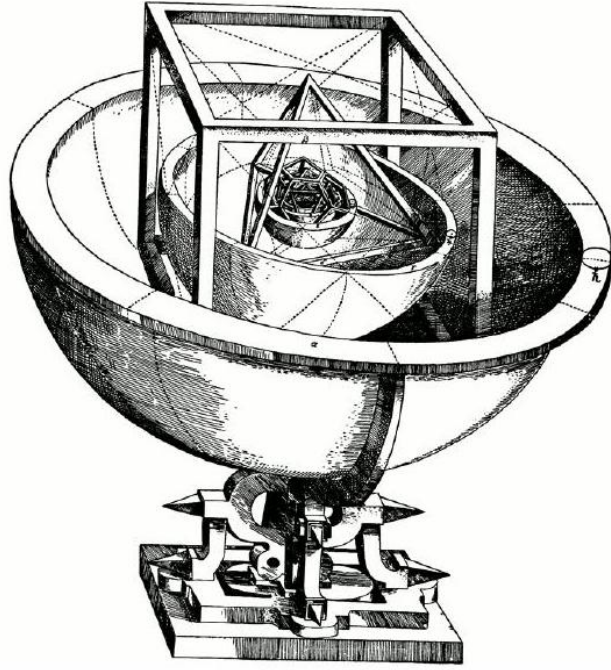
Kepler summarized Tycho Brahe's observational data and proposed three laws of planetary motion, based on which Newton identified the inverse-square law of gravity. This was actually a quite complicated and interesting story.

Kepler knew six planets including Earth: Mercury, Venus, Earth, Mars, Jupiter, and Saturn. He tried to explain their orbital spacing using the five *Platonic solids* (正多面体): tetrahedron (四面体), cube (六面体), octahedron (八面体), dodecahedron (十二面体), and icosahedron (二十面体).

His nested-sphere model placed one solid between each neighboring pair of orbital spheres. From the outside inward:

| Orbital spheres    | Solid between them |
|--------------------|--------------------|
| Saturn and Jupiter | Cube               |
| Jupiter and Mars   | Tetrahedron        |
| Mars and Earth     | Dodecahedron       |
| Earth and Venus    | Icosahedron        |
| Venus and Mercury  | Octahedron         |

Each solid is inscribed in the outer sphere and circumscribed around the inner sphere.



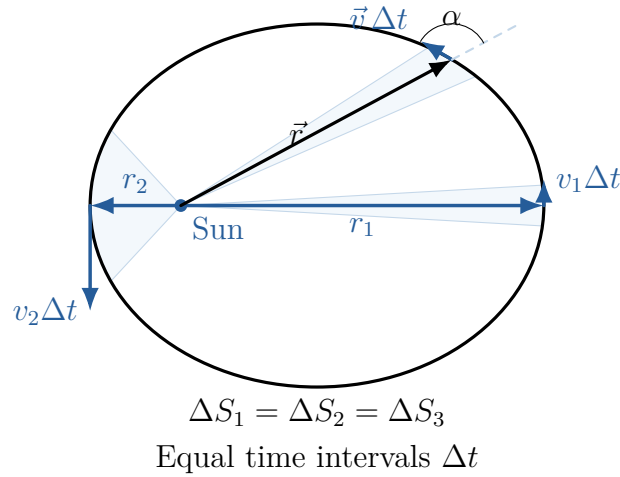
**Fig. 2:** Kepler's nested Platonic-solid model of planetary spacing.

Kepler joined Tycho Brahe (第谷·布拉赫), whose precise observations provided the basis for his study of Mars. Attempts to fit the data with circular orbits exposed persistent discrepancies. Kepler eventually accepted an elliptical orbit; the first two laws were published in 1609, and the third in 1619. His search for geometrical harmony led to a different, observation-driven description of planetary motion.<sup>[1]</sup>

Planetary ellipses can look almost circular. Representative orbital eccentricities (偏心率) are approximately

$$\begin{aligned} e_{\text{Mercury}} &= 0.21, & e_{\text{Venus}} &= 0.007, & e_{\text{Earth}} &= 0.017, \\ e_{\text{Mars}} &= 0.09, & e_{\text{Jupiter}} &= 0.05, & e_{\text{Saturn}} &= 0.06, \\ e_{\text{Uranus}} &= 0.05, & e_{\text{Neptune}} &= 0.009. \end{aligned}$$

The Moon's orbit around Earth has  $e_{\text{Moon}} \approx 0.05$ . Kepler's laws became a foundation for Newton's theory of gravity.



**Fig. 3:** The swept area satisfies  $\Delta S \simeq L\Delta t/(2m)$ .

## 6.2 Kepler's three laws

### 6.2.1 The first law

The planet's orbit is a planar ellipse, and the Sun lies at one of the foci of the ellipse.

Kepler's first law is actually a very strong statement. Generally speaking, a planet moves in three-dimensional (3D) space, but this law states that it is planar. A planar motion does not always form a closed orbit, but Kepler's first law asserts that it is closed and periodic. Copernicus, influenced by the aesthetic philosophy of the Greeks, thought that a planet's orbit should be a circle. Nevertheless, Kepler figured out that, in general, a planetary orbit is an ellipse. A circular orbit is a special case in which the Sun lies at the center.

### 6.2.2 The second law

Since the general orbit of a planet is elliptical rather than circular, the motion at each point on the orbit is different. To connect the motion along the orbit is precisely what Kepler's second law tells us:

*The areas swept by the line connecting the Sun and a planet are equal in equal time intervals.*

Assume that within a time interval  $\Delta t$ , the line running from the Sun to a planet sweeps an area  $\Delta S$ . Kepler's second law states that  $\Delta S/\Delta t$  is a constant.

Consider two special positions in a planetary orbit: aphelion (远日点, the farthest point from the Sun) and perihelion (近日点, the nearest point from the Sun). Denote the displacement vectors relative to the Sun at the aphelion and perihelion as  $\mathbf{r}_1$  and  $\mathbf{r}_2$ , respectively. Correspondingly, the velocities are  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , then  $\mathbf{v}_1 \perp \mathbf{r}_1$  and  $\mathbf{v}_2 \perp \mathbf{r}_2$ .

The arc length traveled around the aphelion during time  $\Delta t$  is  $\Delta s = v_1 \Delta t$ , and the area swept at the aphelion is  $\Delta S = \frac{1}{2} r_1 \Delta s = \frac{1}{2} r_1 v_1 \Delta t$ . Similarly, the same area should be swept during  $\Delta t$  around the perihelion:  $\Delta S = \frac{1}{2} r_2 v_2 \Delta t$ . Then

$$m r_1 v_1 = m r_2 v_2, \quad (6.1)$$

where the planet mass  $m$  is multiplied. The product of linear momentum and displacement is actually the angular momentum, whose precise definition will be given later. It means that the angular momentum  $L_1$  at the aphelion equals  $L_2$  at the perihelion. Since  $r_1 > r_2$ , we have  $v_1 < v_2$ .

If the planet is at a general point in the orbit, then  $\mathbf{v}$  and  $\mathbf{r}$  are no longer perpendicular. Their relative angle is denoted by  $\alpha$ . Then the area swept during a small time interval  $\Delta t$  is

$$\Delta S \simeq \frac{1}{2} r v \sin \alpha \Delta t, \quad (6.2)$$

and, with an area direction,

$$\Delta \mathbf{S} \simeq \frac{1}{2} (\mathbf{r} \times \mathbf{v}) \Delta t.$$

Hence it means angular momentum conservation,

$$\mathbf{L} = m \mathbf{r} \times \mathbf{v}, \quad (6.3)$$

which does not change with time.

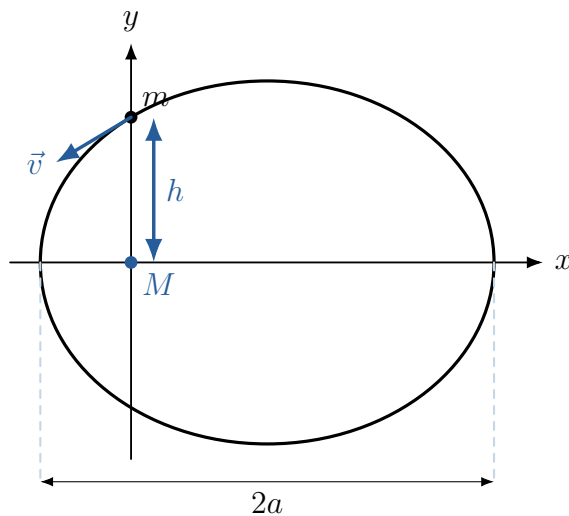
Angular momentum conservation is a fundamental law of nature as a consequence of spatial isotropy, which will be explained later. Simply put, because the gravitational force passes through the Sun's center, it does not generate torque to change the angular momentum.

### 6.2.3 The third law

Different initial conditions can lead to different orbits. Kepler's third law connects different orbits:

*For different orbits, the ratio between the cube of the semi-major axis and the square of the period is a constant.*

For an elliptic orbit (Fig. 4), the origin is set at the focus. The  $y$ -axis intersects the ellipse and cuts a chord (the latus rectum (通径)), whose half-length is denoted as  $h$ . The  $x$ -axis is the major axis intersecting the ellipse, whose half-length is  $a$ . Denote the semi-minor axis by  $b$ ; ellipse geometry gives  $b^2 = ah$ .

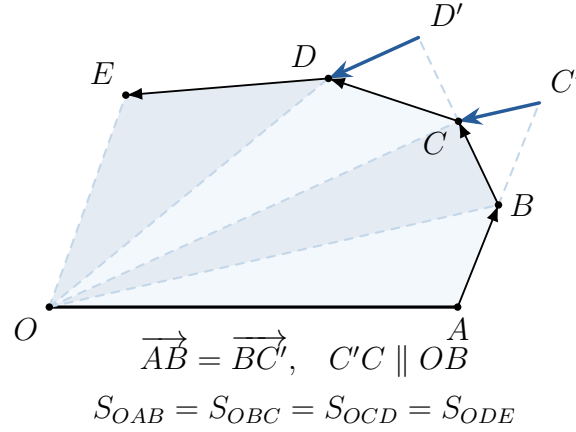


$$\begin{aligned} E &= E_K + E_P \\ &= \frac{mv^2}{2} - \frac{GMm}{r} \\ &= -\frac{GMm}{2a} \end{aligned}$$

$$L = m|\vec{r} \times \vec{v}| = m\sqrt{GMh}$$

**Fig. 4:** The semi-major axis  $a$  determines energy; the semi-latus rectum  $h$  determines angular momentum.





**Fig. 6:** Successive central impulses preserve the swept area in equal time steps.

### 6.3.1 Proof of Kepler's second law

Assume gravity is a central force field, which is sufficient for the proof of Kepler's second law. Feynman's classroom joke imagines angels pushing planets along their paths. The Newtonian question is where the force must point: a central force bends the motion toward the Sun.<sup>[2]</sup>

The geometric picture to prove Kepler's second law is presented in Fig. 6. Suppose the Sun is located at  $O$  and the planet starts from  $A$ . Within a small time interval  $\Delta t$  (first-order infinitesimal), it moves to  $B$ . We use a short line segment  $AB$  to approximate its trajectory, with an error of second order. The velocity is  $\mathbf{v}_{AB} = \frac{\mathbf{AB}}{\Delta t}$ . If there were no gravity, in the next time interval  $\Delta t$ , the planet would continue straight toward  $C'$ , such that  $\mathbf{AB} = \mathbf{BC'}$ . It is obvious that

$$S_{\triangle OAB} = S_{\triangle OBC'}. \quad (6.5)$$

However, gravity pulls the planet back. The attraction is along the direction of  $\mathbf{OB}$ , such that the planet is pulled from  $C'$  to  $C$ , hence  $\vec{C'C} \parallel \vec{OB}$ , and  $\mathbf{v}_{BC} = \frac{\mathbf{BC}}{\Delta t}$ . It is easy to show that

$$S_{\triangle OBC} = S_{\triangle OBC'}, \quad (6.6)$$

since they share the same base and height. Hence within the same time interval  $\Delta t$ , the areas swept by the Sun–planet line are equal,

$$S_{\triangle OAB} = S_{\triangle OBC}. \quad (6.7)$$

Repeating this process — after consecutive intervals  $\Delta t$  the planet arrives at  $D, E, \dots$  — gives

$$S_{\triangle OAB} = S_{\triangle OBC} = S_{\triangle OCD} = S_{\triangle ODE} = \dots \quad (6.8)$$

Moreover, one can see that the trajectory  $ABCD \dots$  is a planar curve. This completes the proof of Kepler's second law.

Let  $\theta$  denote the polar angle of the orbital position, distinct from the angle  $\alpha$  between position and velocity. Choose its positive direction along the motion. In the continuous limit,

$$\frac{dS}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2m}.$$

Here  $L$  is the magnitude of orbital angular momentum. Including the area direction gives

$$2m \frac{d\mathbf{S}}{dt} = m \mathbf{r} \times \mathbf{v} = \mathbf{L}. \quad (6.9)$$

### 6.3.2 Proof of Kepler's first law

Next we prove that a bound, non-radial orbit is an ellipse, using a geometric construction in velocity space.<sup>[3]</sup>

The proof has four steps:

1. Combine equal areas with inverse-square gravity to obtain a circle in velocity space.
2. Use the bound-orbit condition to locate the velocity origin inside this circle.
3. Rotate the diagram and construct an auxiliary ellipse from perpendicular bisectors.
4. Match radial directions and tangents to show that the physical orbit has the same shape.

Kepler's second law alone does not ensure a closed orbit. We consider bound motion with nonzero angular momentum; the construction below will establish the elliptical shape. Assume gravity is a central inverse-square law,

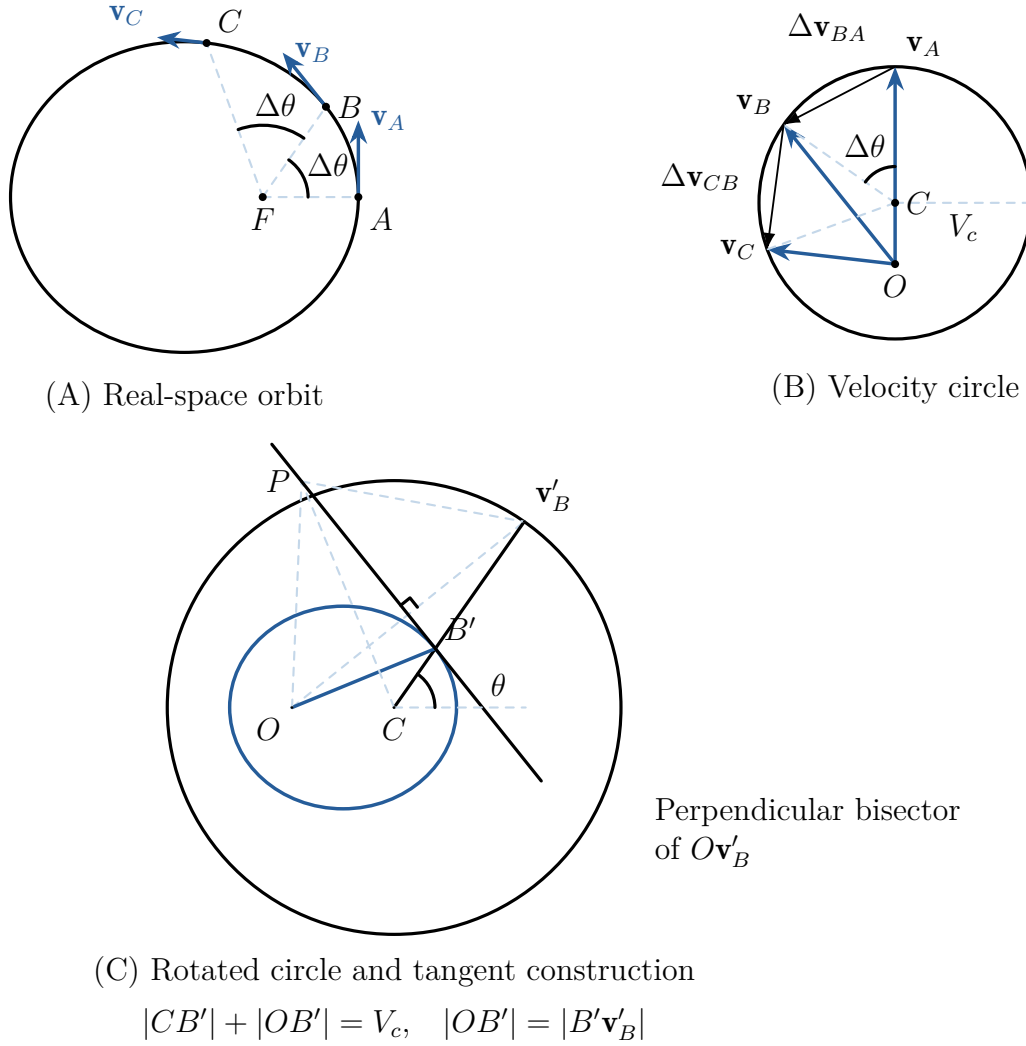
$$\mathbf{F} = -\frac{GMm}{r^2} \mathbf{e}_r, \quad (6.10)$$

where  $M$  is the Sun's mass and  $m$  is the planet's mass. We use the fixed-Sun approximation  $m \ll M$  and neglect other bodies.

Newton realized the inverse-square law by comparing the Moon's orbital motion with free fall on the ground. The Earth–Moon distance is approximately 60 Earth radii. The falling distance in one second on Earth's surface is about 5 m. How much is the Moon's “falling distance” in one second? Without Earth's gravity, the Moon would follow the tangent with distance  $\Delta x = v\Delta t$ . To bend back to a circle, the Moon must “fall” a distance  $s$  satisfying

$$\begin{aligned} \frac{v\Delta t}{s} &\simeq \frac{2d_{em}}{v\Delta t}, \\ s &\simeq \frac{1}{2} \frac{v^2}{d_{em}} (\Delta t)^2 = \frac{1}{2} \omega^2 d_{em} (\Delta t)^2 = \frac{2\pi^2}{T^2} d_{em} (\Delta t)^2. \end{aligned} \quad (6.11)$$

Plugging in  $T = 27.3$  days and  $d_{em} = 60 r_e$ , we arrive at  $s \approx 1.36$  mm, which is about  $1/3676 \approx 1/60^2$  of the terrestrial 5 m. This crude estimate accurately suggests the inverse-square relation, building Newton's confidence that satellites and planets obey the same law. Thus the concept of *universal gravity* was born.



**Fig. 7:** (A) Equal polar-angle steps on the real orbit. (B) The velocity endpoints form an offset circle, the *hodograph* (速度矢端曲线). (C) Rotating the velocity diagram gives a tangent construction of an auxiliary ellipse.

Now prove the elliptic orbit. In Fig. 7A, the Sun is at  $F$ . Start from the perihelion  $A$  on the orbit, and mark points  $B, C, \dots$  such that  $\angle BFA = \angle CFB = \dots = \Delta\theta$  with small  $\Delta\theta$ . The radii are  $r_A, r_B, r_C, \dots$  and the velocities  $\mathbf{v}_A, \mathbf{v}_B, \mathbf{v}_C, \dots$

Compare  $\Delta\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A$  and  $\Delta\mathbf{v}_{CB} = \mathbf{v}_C - \mathbf{v}_B$ . The time to sweep  $\triangle BFA$  is

$$\begin{aligned} \Delta t &\simeq \frac{S_{\triangle BFA}}{dS/dt} \\ &\simeq \frac{\frac{1}{2}r_A^2\Delta\theta}{L/(2m)} = \frac{mr_A^2\Delta\theta}{L}. \end{aligned} \quad (6.12)$$

Hence, by Newton's second law,

$$\Delta\mathbf{v}_{BA} \simeq \frac{\mathbf{F}\Delta t}{m} = -\frac{GMm\Delta\theta}{L}\mathbf{e}_A, \quad (6.13)$$

which points radially inward. The radius dependence cancels. In the limit  $\Delta\theta \rightarrow 0$ , the velocity change per unit polar angle has constant magnitude and turns with  $\theta$ ; the velocity endpoints therefore trace a circle. The tangent to this velocity circle points along the

inward radial direction of the physical orbit. It must be distinguished from the tangent to the physical orbit, which points along  $\mathbf{v}$ .

The circle's center is generally not at the origin  $O$ ; denote it  $C$ . By planar geometry, the angle of  $\Delta\mathbf{v}$  about  $C$  equals its incremental direction change each step. The circle radius is

$$V_c = \left| \frac{d\mathbf{v}}{d\theta} \right| = \frac{GMm}{L}. \quad (6.14)$$

Here  $V_c$  is a speed scale: the radius of the velocity circle, not the radial component of velocity. Since  $d\mathbf{e}_\theta/d\theta = -\mathbf{e}_r$ , Eq. (6.13) gives

$$\frac{d}{d\theta}(\mathbf{v} - V_c\mathbf{e}_\theta) = 0.$$

Thus

$$\mathbf{v} = \mathbf{w} + V_c\mathbf{e}_\theta, \quad \mathbf{w} = \overrightarrow{OC} = \text{constant}.$$

The transverse speed is  $v_\theta = L/(mr)$ , so

$$V_cv_\theta = \frac{GM}{r}, \quad |\mathbf{w}|^2 = v^2 - 2V_cv_\theta + V_c^2.$$

Consequently the orbital energy is

$$\begin{aligned} E &= \frac{1}{2}mv^2 - \frac{GMm}{r} \\ &= \frac{m}{2}(|OC|^2 - V_c^2). \end{aligned}$$

Bound motion has  $E < 0$  when potential energy is zero at infinity. Therefore  $|OC| < V_c$ : the velocity origin  $O$  lies inside the circle.

Rotate the entire velocity diagram clockwise by  $90^\circ$  (Fig. 7C). The circle radius  $C\mathbf{v}'_B$ , originally along  $\mathbf{e}_\theta$ , now points along  $\mathbf{e}_r$ , parallel to the physical radius  $FB$ . Meanwhile, the rotated velocity  $\mathbf{v}'_B$  is perpendicular to the physical tangent  $\mathbf{v}_B$ .

Construct the perpendicular bisector (垂直平分线) of the segment from  $O$  to the endpoint of  $\mathbf{v}'_B$ . It is parallel to  $\mathbf{v}_B$ . As the endpoint runs around the circle, these bisectors envelope an ellipse with foci  $O$  and  $C$  and full major-axis length  $V_c$ . This auxiliary ellipse uses velocity-diagram units; a constant scale factor will relate it to the physical orbit.

To see this, recall the optical property of ellipses: a ray from one focus reflects to the other. Connect  $Cv'_B$ . The perpendicular bisector of  $Ov'_B$  intersects  $Cv'_B$  at  $B'$ . Connect  $OB'$  and  $CB'$ . Since  $O$  and  $v'_B$  are mirror images about the bisector, the ray  $OB'$  reflects to  $C$ , and  $B'v'_B$  is the image of  $OB'$ . Hence

$$|CB'| + |OB'| = |Cv'_B| = \frac{GMm}{L}, \quad (6.15)$$

so  $B'$  lies on the ellipse. Moreover the bisector is tangent to the ellipse: for any other point  $P$  on this bisector,

$$|CP| + |OP| = |CP| + |v'_B P| > |Cv'_B|. \quad (6.16)$$

We now build a point-to-point mapping between the constructed ellipse (Fig. 7C) and the planetary trajectory by using polar coordinates. For the real-space trajectory, set the origin

at  $F$ ; for the constructed ellipse, set the origin at the focus  $C$ . The radius  $CB'$  lies along  $C\mathbf{v}'_B$ , hence parallel to  $FB$ . The auxiliary tangent is the perpendicular bisector of  $O\mathbf{v}'_B$ , hence parallel to  $\mathbf{v}_B$ . Thus at the same polar angle  $\theta$ , the two curves have parallel tangents. For a curve  $\mathbf{r}(\theta) = r \mathbf{e}_r$ ,

$$\frac{d\mathbf{r}}{d\theta} = \frac{dr}{d\theta} \mathbf{e}_r + r \mathbf{e}_\theta. \quad (6.17)$$

The velocity is  $\dot{\mathbf{r}} = \dot{\theta} d\mathbf{r}/d\theta$ . For the two curves,

$$\mathbf{r}_1 = r_1(\theta) \mathbf{e}_r, \quad \mathbf{r}_2 = r_2(\theta) \mathbf{e}_r, \quad (6.18)$$

parallel tangents at the same  $\theta$  imply

$$\frac{1}{r_1} \frac{dr_1}{d\theta} = \frac{1}{r_2} \frac{dr_2}{d\theta}, \quad (6.19)$$

hence

$$\mathbf{r}_1(\theta) = C_0 \mathbf{r}_2(\theta), \quad (6.20)$$

for a constant  $C_0$ . Therefore the real trajectory is similar to the constructed ellipse — i.e., it is an ellipse.

## 6.4 Comment on Kepler's third law

The geometric proof is beautiful and reveals the nature of planetary motion. Next we use calculus for further exploration.

Kepler's third law can be shown by a scaling method. Suppose  $\mathbf{r}(t)$  solves

$$\frac{d^2\mathbf{r}(t)}{dt^2} = -\frac{GM}{r^2} \mathbf{e}_r. \quad (6.21)$$

Perform the scaling

$$\mathbf{r}^s(t) = \lambda_1 \mathbf{r}(\lambda_2 t). \quad (6.22)$$

It is easy to show

$$\begin{aligned} \ddot{\mathbf{r}}^s(t) + \frac{GM}{|\mathbf{r}^s(t)|^2} \mathbf{e}_r &= \lambda_1 \lambda_2^2 \ddot{\mathbf{r}}(\lambda_2 t) + \lambda_1^{-2} \frac{GM}{|\mathbf{r}(\lambda_2 t)|^2} \mathbf{e}_r \\ &= 0, \end{aligned} \quad (6.23)$$

provided

$$\lambda_2^2 \lambda_1^3 = 1. \quad (6.24)$$

Thus the spatial size  $\ell$  of the orbit and the period  $T$  obey

$$\frac{\ell^3}{T^2} = \text{const.} \quad (6.25)$$

Scaling relates geometrically similar orbits. Kepler's third law is stronger: for planets orbiting the same dominant mass  $M$ ,

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM},$$

independent of eccentricity. To establish this stronger statement, divide the ellipse area by the constant areal velocity:

$$\begin{aligned} T &= \frac{\pi ab}{L/(2m)} \\ &= \frac{2\pi mab}{L} \\ &= \frac{2\pi ab}{\sqrt{GMh}}. \end{aligned}$$

Here  $L = m\sqrt{GMh}$  follows by evaluating the velocity at  $\theta = \pi/2$ , measured from perihelion. At perihelion both  $\mathbf{v}$  and  $\mathbf{e}_\theta$  are transverse, so  $\mathbf{w}$  is perpendicular to the major axis. At  $\theta = \pi/2$  its transverse component therefore vanishes:  $v_\theta = V_c$  and  $L = mhV_c$ . With  $V_c = GMm/L$ , this gives  $L^2 = GMm^2h$ . Finally, using  $b^2 = ah$ ,

$$T^2 = \frac{4\pi^2 a^2 b^2}{GMh} = \frac{4\pi^2 a^3}{GM}.$$

We have used a frame centered on the Sun. Next we examine how Newton's equations change when the observer translates, rotates, or accelerates.

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- [4] Historical illustrations of Kepler's nested-solid model and Newton's *Principia* are reproduced from the supplied 2025 Lecture 6 source. The original images are preserved; the teaching diagrams are vector reconstructions.