

Lecture 5: Newton's laws of motion (II): oscillation

From drag to oscillation. In L4, linear air resistance gives $m\ddot{x} = -b\dot{x}$ for horizontal motion. Add a spring: its restoring force (恢复力) $-kx$ pulls the mass toward equilibrium, while drag removes energy. Their competition leads to damped oscillation.

$$\underbrace{m\ddot{x}}_{\text{inertia}} + \underbrace{b\dot{x}}_{\text{damping}} + \underbrace{kx}_{\text{restoring force}} = 0.$$

An external periodic force can continually supply energy. We will follow this progression from free oscillation to damping, driving, and resonance.

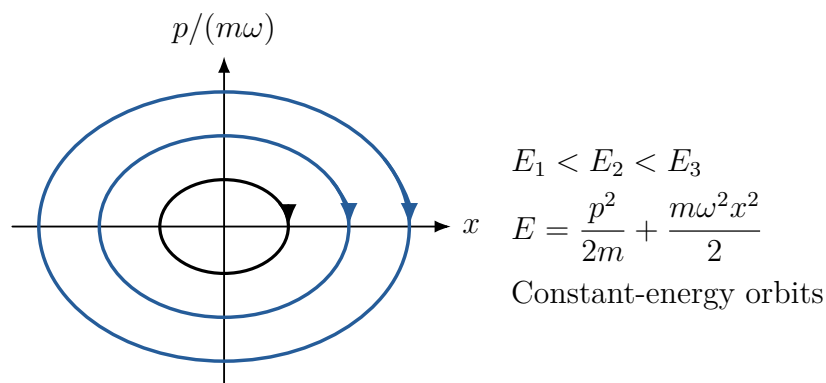


Fig. 1: Phase-space portrait of the harmonic oscillator.

Outline:

1. Harmonic oscillator and uniform circular motion

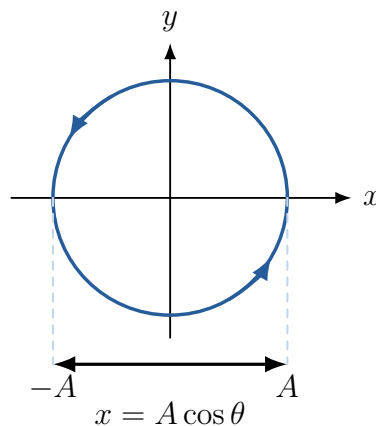


Fig. 2: Projection of uniform circular motion gives harmonic oscillation.

2. Phase space orbit, Bohr–Sommerfeld condition

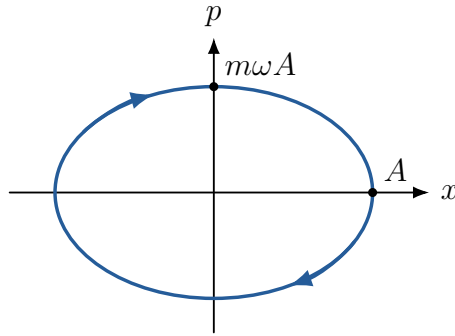


Fig. 3: The oscillator traces an ellipse in phase space.

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$$

$$\oint p dx = \left(n + \frac{1}{2}\right) h$$

3. Damped harmonic oscillators (overdamped, underdamped, critical)

$$\ddot{x} + \frac{1}{\tau}\dot{x} + \omega_0^2 x = 0$$

$$x(t) = A \cos(\omega t + \phi) e^{-\beta t}$$

$$\omega_d = \omega_0 \sqrt{1 - \left(\frac{1}{2\omega_0\tau}\right)^2}, \quad \beta = \frac{1}{2\tau}$$

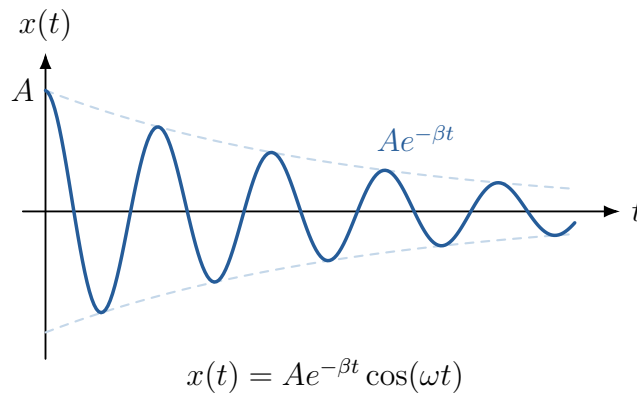


Fig. 4: Damped oscillation and its exponential envelope.

4. Cyclotron orbit:

$$R = \sqrt{\frac{\hbar c}{qB}}$$

5. LC oscillators

$$L \cdot \frac{d^2 Q}{dt^2} + R \cdot \frac{dQ}{dt} + \frac{Q}{C} = 0.$$

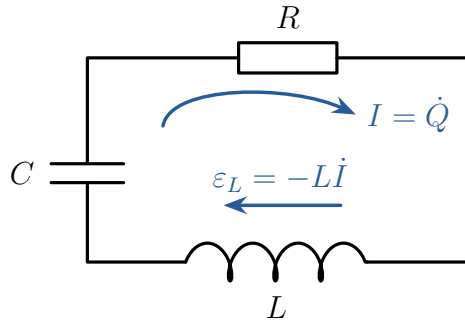


Fig. 5: An RLC circuit: an electrical example of damped oscillation.

5.1 Mathematical tools for oscillations

We start in the middle: assume integers, counting, and the idea of *add one*. From there we keep the *rules* and enlarge the *numbers* only when a problem can't be solved without doing so.^[1]

1. Addition $\Rightarrow$ multiplication $\Rightarrow$ powers. Define $a + b$, then ab as repeated addition, then a^n as repeated multiplication. The payoff is the simple algebra of exponents:

$$a^{m+n} = a^m a^n, \quad (a^m)^n = a^{mn}.$$

2. Inverses. Solve the direct operations backwards: subtraction, division, roots, logarithms. Logs turn products into sums:

$$\log(ab) = \log a + \log b, \quad e^{x+y} = e^x e^y.$$

3. Generalize the numbers. When an equation has no solution, enlarge the number system but keep the rules:

$$\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{R} \rightarrow \mathbb{C}, \quad i^2 = -1.$$

Complex numbers close algebraic equations and, miraculously, we don't need to invent anything beyond them.

4. Euler's formula. Imaginary powers define the bridge from algebra to geometry:

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Hence real oscillations are just complex exponentials seen from the real axis.

5. Differential equations become algebra. Exponentials are eigenfunctions of differentiation: if $x(t) = e^{rt}$, then $Dx = rx$ and $D^2x = r^2x$. A linear constant-coefficient ODE,

$$x'' + 2\gamma x' + \omega_0^2 x = 0,$$

reduces to the *characteristic polynomial*

$$r^2 + 2\gamma r + \omega_0^2 = 0.$$

For underdamped motion ($\gamma < \omega_0$), roots $\Rightarrow$ solution form, with $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$:

$$r = -\gamma \pm i\omega_d,$$

$$x(t) = e^{-\gamma t} [A \cos(\omega_d t) + B \sin(\omega_d t)].$$

These tools turn oscillation equations into algebraic problems; a mathematics course develops their foundations in greater depth.

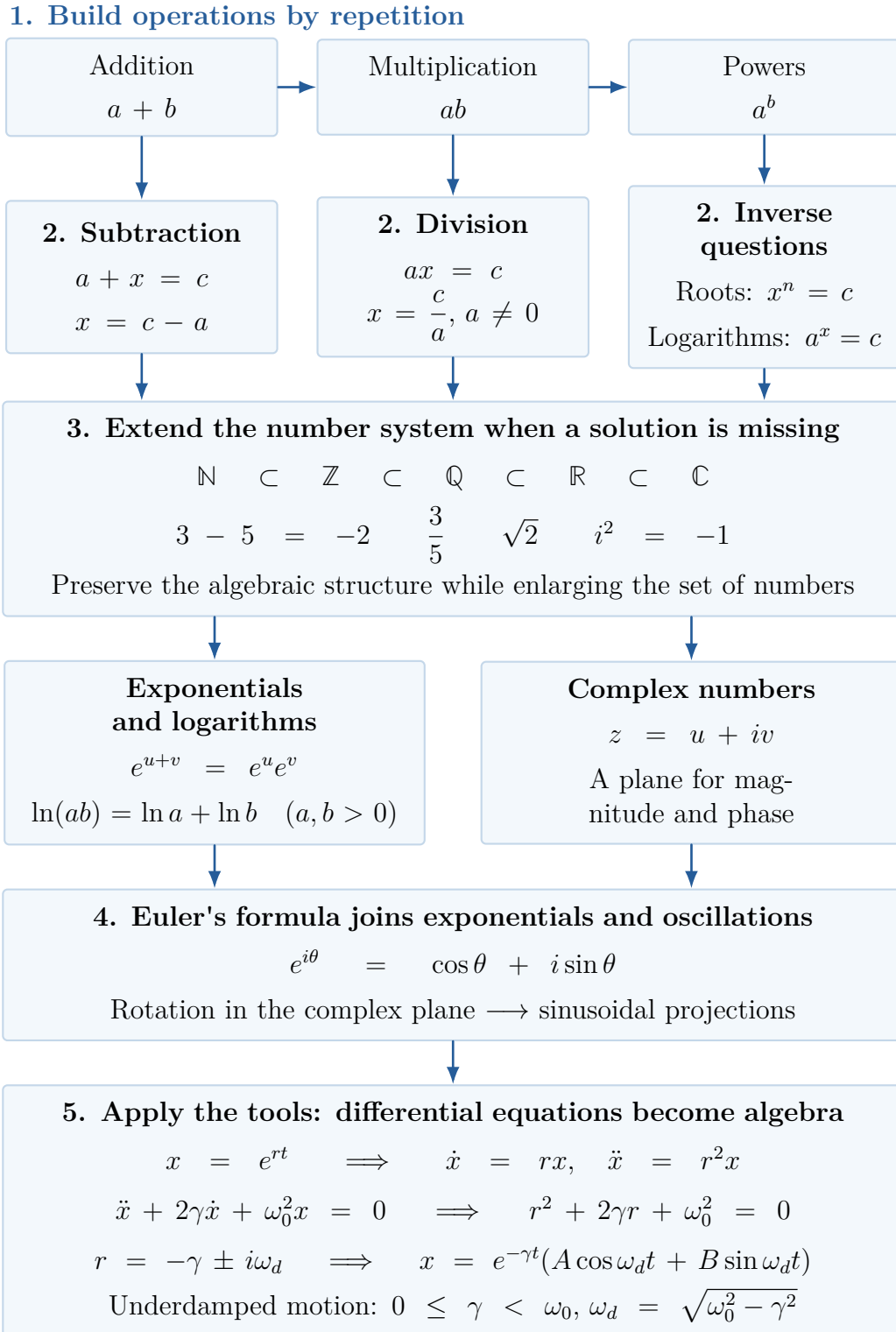


Fig. 6: The logic of Section 5.1: operations, inverse questions, number systems, and Euler's formula lead to tools for oscillations.^[1]

5.2 Harmonic oscillators

Oscillations are a general class of phenomena, such as the elastic spring oscillator, simple pendulum, and the electromagnetic analogy of LC oscillator, etc. Quantum mechanically, each quantum mode in the free case can be viewed as a harmonic oscillator, such as the photon mode, the lattice vibration — phonon modes, etc. Waves can be viewed as a series of vibration modes propagating in spacetime, including mechanical wave, E&M wave, and quantum mechanical, gravitational waves.

Harmonic oscillator is a prototype system to study in all branches of physics. It is simple, elegant, both in a classical way and in the quantum way. Harmonic oscillator is also the first QM problem solved by Heisenberg.

According to Hooke's law, the restoring force

$$F = -kx,$$

where x is measured from its equilibrium position. Then

$$m \frac{d^2 x}{dt^2} = -kx$$

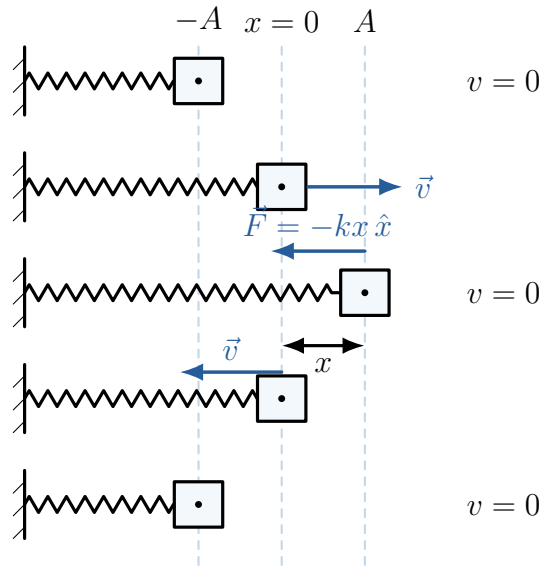


Fig. 7: A spring provides a restoring force toward equilibrium.

This is a second-order constant-coefficient differential equation. Actually, we could find an elementary but smart method to solve it!

Consider a uniform circular motion, with the radius A and circular frequency ω . The acceleration

$$\begin{aligned} \vec{a} &= -a\hat{e}_r = -a(\cos\theta\hat{x} + \sin\theta\hat{y}) \\ &= -\frac{a}{A}(x\hat{x} + y\hat{y}) \\ &= -\omega^2(x\hat{x} + y\hat{y}) \end{aligned}$$

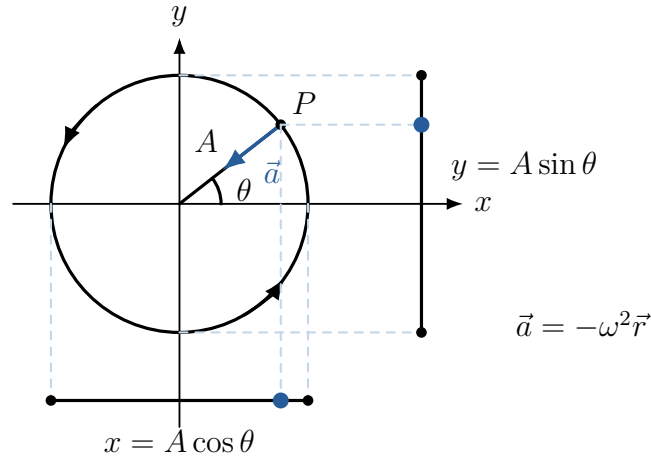


Fig. 8: Projecting the centripetal acceleration onto the x axis.

Projecting onto the x axis gives

$$a_x = \frac{d^2x}{dt^2} = -\omega^2 x.$$

This is the harmonic-oscillator equation when

$$\omega = \sqrt{\frac{k}{m}}.$$

The solution to the uniform circular motion is obvious $\theta = \omega t$. Hence

$$x = A \cos(\omega t + \varphi),$$

$$y = A \sin(\omega t + \varphi).$$

where A and φ are integration constants determined by the initial conditions.

$$x(0) = A \cos \varphi,$$

$$\dot{x}(0) = -\omega A \sin \varphi.$$

Thus

$$A^2 = x^2(0) + \left(\frac{\dot{x}(0)}{\omega} \right)^2,$$

$$\tan \varphi = -\frac{\dot{x}(0)}{\omega x(0)}.$$

The signs of $x(0)$ and $\dot{x}(0)$ fix the quadrant of φ .

Generally speaking, this kind of differential equation can be solved by trying $x(t) = e^{\lambda t}$, plugging in $\ddot{x} = -\omega^2 x \Rightarrow \lambda^2 = -\omega^2 \Rightarrow \lambda = \pm i\omega$. Hence,

$$x(t) = a_1 e^{i\omega t} + a_2 e^{-i\omega t}$$

If we require $x(t)$ is real, then $a_1 = a_2^* = \frac{A}{2} e^{i\varphi}$, then $\Rightarrow x(t) = A \cos(\omega t + \varphi)$.

$$v(t) = \dot{x}(t) = -A\omega \sin(\omega t + \varphi)$$

We have seen the relationship between uniform circular motion and harmonic oscillation. Actually, the relation is even closer if we organize

$$-\frac{v(t)}{A\omega} = \sin(\omega t + \varphi) = \frac{y(t)}{A}$$

Here $y(t)$ is the auxiliary coordinate in the circular-motion picture, so $y(t) = -v(t)/\omega$. Hence, the y -axis motion in fact reflects *the velocity of the oscillation!*

5.2.1 Phase-space orbits

In the canonical version of classical mechanics, momentum $\vec{p} = m\vec{v}$ is also a fundamental quantity. Based on the above reasoning,

$$\begin{cases} X = A \cos(\omega t + \varphi) \\ P = mv = -mA\omega \sin(\omega t + \varphi) \end{cases}$$

$$\Rightarrow \left(\frac{X}{A}\right)^2 + \left(\frac{P}{mA\omega}\right)^2 = 1$$

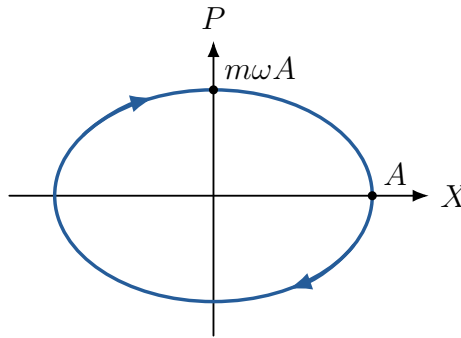


Fig. 9: The harmonic oscillator follows a directed ellipse in phase space.

Please note that the phase space orbit is chiral, i.e., it only rotates in one direction, but not in the reverse direction!

cf. *the cyclotron motion of electrons in the magnetic field.*

Classically, the area enclosed by the orbit is arbitrary. Quantum mechanically, however, the area has a minimum $\frac{h}{2}$.

Bohr-Sommerfeld quantization condition

$$\oint p dx = \left(n + \frac{1}{2}\right) h, \quad n = 0, 1, 2, \dots$$

over a period

Minimal area:

$$\pi \cdot A_{\min}^2 \omega m = \frac{h}{2}$$

$$\Rightarrow l_0 = A_{\min} = \sqrt{\frac{\hbar}{m\omega}}$$

Hence, for a harmonic oscillation, there exists a minimum length scale l_0 .

5.2.2 Uncertainty principle

Since the orbit area has a minimum, it means oscillators cannot be at rest. The uncertainty $\sqrt{\langle(\Delta x)^2\rangle} \sqrt{\langle(\Delta p)^2\rangle}$ is of the same order as the area of the minimum orbit. Roughly speaking

$$\sqrt{\langle(\Delta x)^2\rangle} \sqrt{\langle(\Delta p)^2\rangle} \sim \hbar$$

a more precise calculation based on quantum mechanics shows

$$\sqrt{\langle(\Delta x)^2\rangle} \sqrt{\langle(\Delta p)^2\rangle} \geq \frac{\hbar}{2}$$

5.2.3 Conservation of energy

The phase-space ellipse is not only a trajectory; it is a constant-energy curve. This is the first appearance of conservation-law thinking in this lecture. We have seen

$$\left(\frac{p}{mA\omega}\right)^2 + \left(\frac{x}{A}\right)^2 = 1.$$

Multiplying by $\frac{1}{2}m\omega^2 A^2$ gives

$$\underbrace{\frac{p^2}{2m}}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}m\omega^2 x^2}_{\text{potential energy}} = \underbrace{\frac{1}{2}m\omega^2 A^2}_{\text{total energy}}.$$

The first term is the kinetic energy: $\frac{p^2}{2m} = \frac{1}{2}mv^2$.

The second term is determined by the status (location x) of the spring.

Let's calculate the work done to change the spring length to x :

$$\begin{aligned} W_{\text{ext}} &= \int_0^x F_{\text{ext}} dx' = k \int_0^x x' dx' \\ &= \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2. \end{aligned}$$

Here $F_{\text{ext}} = kx'$ is the force required to stretch the spring quasistatically.

5.3 Damped harmonic oscillators

Consider a harmonic oscillator, but with linear resistance.

$$m\ddot{x} = -m\omega_0^2 x - b\dot{x},$$

$$\ddot{x} + \frac{1}{\tau}\dot{x} + \omega_0^2 x = 0.$$

where $\tau = m/b$.

5.3.1 Underdamped motion

Trying an exponential solution: $x(t) \sim e^{i\lambda t} \Rightarrow -\lambda^2 + \frac{i\lambda}{\tau} + \omega_0^2 = 0$

$$\lambda_{1,2} = \pm \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2} + \frac{i}{2\tau}$$

Hence,

$$x(t) = (a_1 e^{i\omega t} + a_2 e^{-i\omega t}) e^{-t/(2\tau)},$$

$$\omega = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2},$$

$$\beta = \frac{1}{2\tau}.$$

$\Rightarrow$ friction decreases frequency

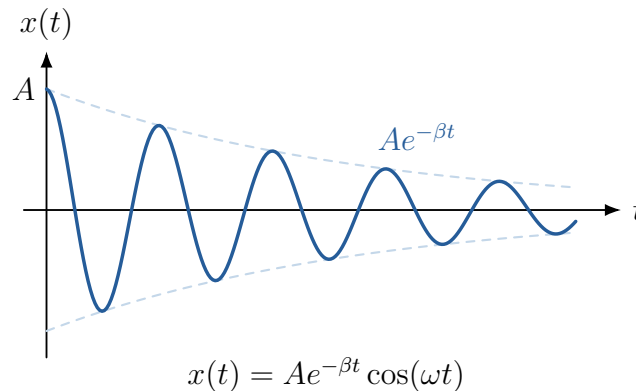


Fig. 10: Underdamped motion has a decaying oscillation envelope.

Consider the real solution:

$$x(t) = A \cos(\omega t + \phi) e^{-\beta t},$$

$$\omega = \omega_0 \sqrt{1 - \left(\frac{1}{2\omega_0\tau}\right)^2}.$$

The above solution works at $\frac{1}{2\tau} < \omega_0$, i.e., $\omega_0\tau > 1/2$. This situation is called the **underdamped** case.

Suppose we know the initial condition x_0 and v_0 at $t=0$, how to determine a_1, a_2 ?

Choose $a_2 = a_1^*$ and write $a_1 = a_R + ia_I$. Then

$$x_0 = a_1 + a_1^* = 2a_R,$$

$$\dot{x}(t) = i\omega (a_1 e^{i\omega t} - a_1^* e^{-i\omega t}) e^{-t/(2\tau)} - \frac{x(t)}{2\tau},$$

$$v_0 = i\omega (a_1 - a_1^*) - \frac{a_1 + a_1^*}{2\tau}$$

$$= -2\omega a_I - \frac{a_R}{\tau}.$$

Therefore

$$a_R = \frac{x_0}{2},$$

$$a_I = -\frac{v_0 + \frac{x_0}{2\tau}}{2\omega}.$$

$$x(t) = e^{-t/(2\tau)} \left[x_0 \cos \omega t + \frac{v_0 + \frac{x_0}{2\tau}}{\omega} \sin \omega t \right].$$

The *quality factor* (品质因数) is $Q = \omega_0 \tau$. For weak damping, the energy decays by a factor e in approximately $Q/(2\pi)$ oscillation periods.

5.3.2 Overdamped motion

When $\omega_0 \tau < \frac{1}{2}$, the quantity $\omega_0^2 - (2\tau)^{-2}$ is negative and the motion has two real decay rates. Try $x(t) \sim e^{-\lambda t}$, $\lambda^2 - \lambda/\tau + \omega_0^2 = 0$

$$\lambda_{1,2} = \frac{1}{2\tau} \pm \sqrt{\left(\frac{1}{2\tau}\right)^2 - \omega_0^2}$$

$$X(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t}$$

The initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ give

$$a_1 = -\frac{v_0 + \lambda_2 x_0}{\lambda_1 - \lambda_2},$$

$$a_2 = \frac{v_0 + \lambda_1 x_0}{\lambda_1 - \lambda_2}.$$

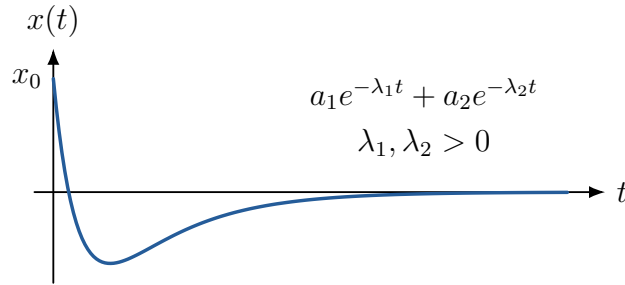


Fig. 11: Overdamped motion is a sum of two decaying exponentials.

This situation is called the **overdamped** case.

5.3.3 Critical damping

Here $\omega_0 \tau = \frac{1}{2}$ and $\lambda = \frac{1}{2\tau}$.

$$x(t) = (a_1 + a_2 t)e^{-\lambda t},$$

$$x_0 = a_1,$$

$$v_0 = -\lambda a_1 + a_2,$$

$$a_2 = v_0 + \frac{x_0}{2\tau}.$$

Hence

$$x(t) = \left[x_0 + \left(v_0 + \frac{x_0}{2\tau} \right) t \right] e^{-t/(2\tau)}.$$

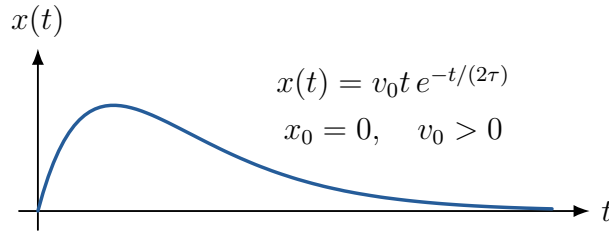
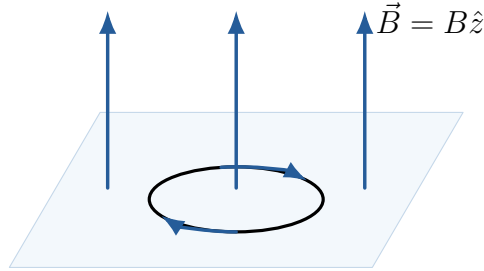


Fig. 12: Critical damping with $x_0 = 0$.

5.4 Other examples of oscillations

5.4.1 Motion in a uniform magnetic field

$$\vec{F} = \frac{q}{c} \vec{v} \times \vec{B}. \quad (\text{Gaussian unit}) \quad \text{set } \vec{B} = B\hat{z}$$



Orbit plane $\perp \vec{B}$
 $q > 0, \quad v_z = 0$

Fig. 13: A circular orbit in the plane perpendicular to the magnetic field.

$$\begin{cases} m\dot{v}_x = \frac{q}{c} B v_y \\ m\dot{v}_y = -\frac{qB}{c} v_x \\ m\dot{v}_z = 0 \end{cases}$$

The signed cyclotron frequency is

$$\omega_c = \frac{qB}{mc}.$$

$$\dot{v}_x = \omega_c v_y,$$

$$\dot{v}_y = -\omega_c v_x.$$

Combining the two components,

$$\dot{v}_x + i\dot{v}_y = -i\omega_c(v_x + iv_y).$$

Define $\eta = v_x + iv_y \Rightarrow \dot{\eta} = -i\omega_c\eta$.

$$\Rightarrow \eta = Ae^{-i\omega_c t} \text{ with } A = v_x(0) + iv_y(0).$$

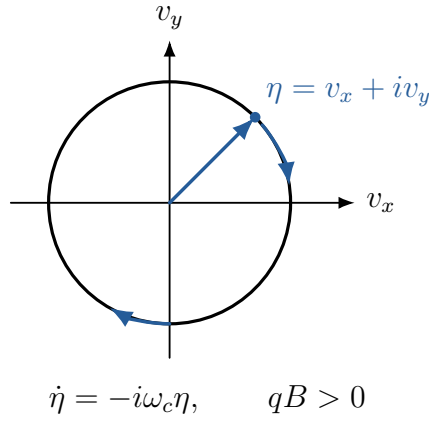


Fig. 14: The velocity-space circle; the sign of qB fixes the rotation direction.

We also define $z = x + iy$.

$$\begin{aligned} \Rightarrow z &= \text{const.} + \int^t \eta dt \\ &= z_0(\text{center of circle}) + \frac{iA}{\omega_c} e^{-i\omega_c t}. \end{aligned}$$

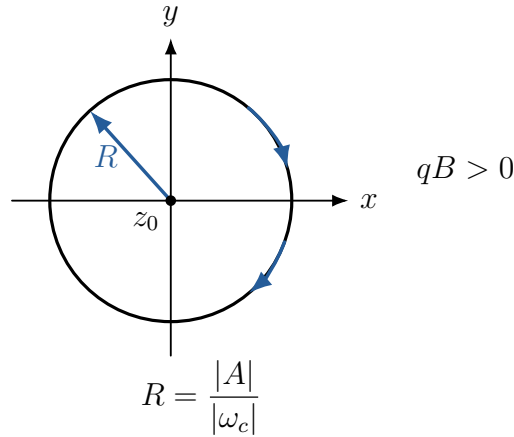


Fig. 15: The real-space orbit is centered at z_0 , with radius $|A|/|\omega_c|$.

The cyclotron radius is

$$R = \left| \frac{A}{\omega_c} \right| = \frac{mvc}{|q|B}.$$

$$P = mv = \frac{|q|BR}{c}, \quad |L| = PR = \frac{|q|BR^2}{c}.$$

Classically, the circular orbit can be of any size.
 But quantum mechanically, it has a minimal size.
 The orbital angular momentum satisfies

$$|L|_{\min} = \frac{|q|BR^2}{c} = \hbar,$$

$$R = \sqrt{\frac{\hbar c}{|q|B}}.$$

5.4.2 LC oscillators

Choose $I = \dot{Q}$ and a loop orientation such that the driven circuit obeys $L\dot{I} + RI + Q/C = \varepsilon(t)$. The free oscillator is the special case $\varepsilon(t) = 0$.

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = \varepsilon(t)$$

$$\Rightarrow L\frac{dI}{dt} + IR + \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$$

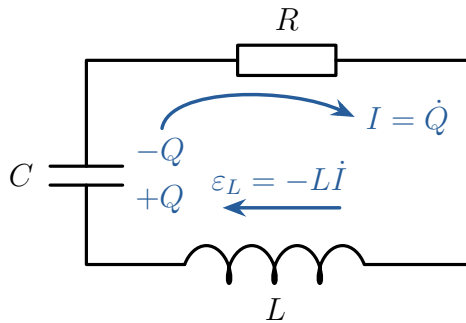


Fig. 16: An LC circuit provides an electrical analogue of a mechanical oscillator.

The analogy to momentum P follows the mapping:

$$\begin{array}{lll} P = mv & \rightarrow & LI = \Phi \\ x & \rightarrow & Q \\ \dot{x} & \rightarrow & I \end{array}$$

We arrived at the harmonic oscillator in an electric circuit

$$\left\{ \begin{array}{l} L \rightarrow m, \frac{1}{C} \rightarrow k \\ \frac{R}{L} = \frac{1}{\tau} \Rightarrow \omega_0^2 = k/m = \frac{1}{LC} \end{array} \right.$$

$$\left(\frac{x}{A}\right)^2 + \left(\frac{P}{m\omega A}\right)^2 = 1$$

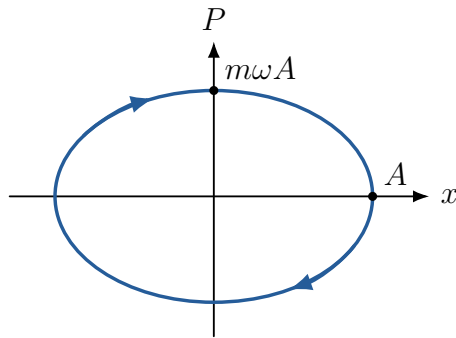


Fig. 17: The mechanical oscillator's phase-space ellipse.

$$\left(\frac{Q}{Q_0}\right)^2 + \left(\frac{\Phi}{LQ_0\omega_0}\right)^2 = 1$$

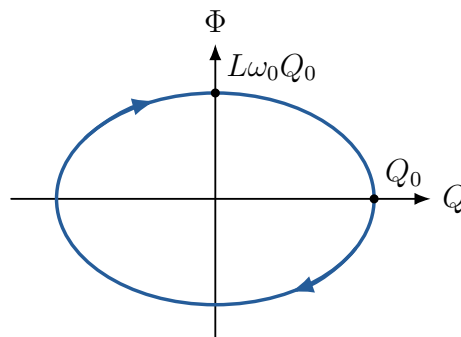


Fig. 18: An LC oscillator exchanges electric and magnetic energy.

Magnetic energy $\rightleftharpoons$ electric energy

5.5 Driven damped oscillations

Consider the LC circuit: in addition to the inductor, capacitor, and resistor, there is also a driving EMF $\varepsilon(t)$.

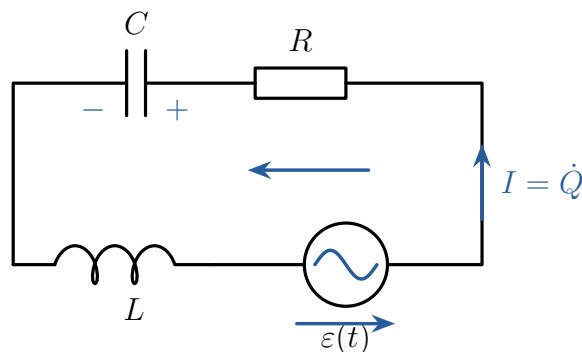


Fig. 19: A driven circuit includes a time-dependent electromotive force.

The governing equations are

$$\begin{aligned} L \frac{dI}{dt} + RI + \frac{Q}{C} &= \varepsilon(t), \\ L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} &= \varepsilon(t), \\ \frac{1}{\tau} &\equiv \frac{R}{L}, \quad \omega_0^2 \equiv \frac{1}{LC}. \end{aligned}$$

For a mechanical oscillator (mass m and external force F),

$$\ddot{x} + \frac{1}{\tau} \dot{x} + \omega_0^2 x = f(t), \quad f(t) \equiv \frac{F(t)}{m}.$$

These are the same constant-coefficient, linear, inhomogeneous ODEs. By superposition,

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t)$ solves the homogeneous equation

$$\ddot{x}_h + \frac{1}{\tau} \dot{x}_h + \omega_0^2 x_h = 0$$

and $x_p(t)$ is any particular solution of

$$\ddot{x}_p + \frac{1}{\tau} \dot{x}_p + \omega_0^2 x_p = f(t) \quad \leftarrow \text{steady solution.}$$

Because the homogeneous part decays in time, the long-time behavior is determined by $x_p(t)$.

5.5.1 Sinusoidal drive

Take $f(t) = f_0 \cos \omega t$. Promote to the complex field by also considering

$$\ddot{y} + \frac{1}{\tau} \dot{y} + \omega_0^2 y = f_0 \sin \omega t, \quad \text{and define } z = x + iy.$$

Then z obeys

$$\ddot{z} + \frac{1}{\tau} \dot{z} + \omega_0^2 z = f_0 e^{i\omega t}.$$

Try a particular solution $z_p(t) = C e^{i\omega t}$. Substituting gives

$$\begin{aligned} C \left[-\omega^2 + \frac{i\omega}{\tau} + \omega_0^2 \right] &= f_0, \\ \Rightarrow C &= \frac{f_0}{\omega_0^2 - \omega^2 + i\omega/\tau}. \end{aligned}$$

Write $C = A e^{-i\delta}$ with

$$\begin{aligned} A^2 &= \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2}, \\ \tan \delta &= \frac{\frac{\omega}{\tau}}{\omega_0^2 - \omega^2}, \quad 0 \leq \delta \leq \pi. \end{aligned}$$

Hence

$$x_p(t) = \Re[Ae^{i(\omega t - \delta)}] = A \cos(\omega t - \delta).$$

The full solution is

$$x(t) = A \cos(\omega t - \delta) + \underbrace{c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}_{\text{transient}},$$

which decays to the steady motion as $t \rightarrow \infty$. For a weakly damped system ($1/\tau \ll \omega_0$),

$$x(t) = A \cos(\omega t - \delta) + e^{-t/(2\tau)} [B_1 \cos(\omega_d t) + B_2 \sin(\omega_d t)],$$

where the damped natural frequency is

$$\omega_d = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2}.$$

Transient motion depends on initial conditions but decays; different initial conditions lead to the same steady motion (attractor).

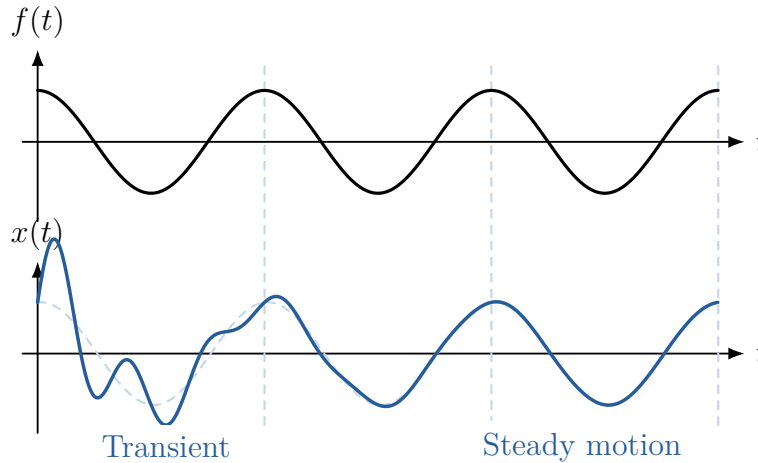


Fig. 20: The transient decays, leaving a steady oscillation at the driving frequency.

5.6 Resonance of a driven oscillator

Under the driving force $f(t) = f_0 e^{i\omega t}$ we have

$$z_p(t) = \frac{f_0 e^{i\omega t}}{\omega_0^2 - \omega^2 + i\omega/\tau}.$$

Asymptotic limits.

$$\omega \rightarrow 0 : \quad z_p(t) \rightarrow \frac{f_0}{\omega_0^2} e^{i\omega t},$$

$$\omega \rightarrow \infty : \quad z_p(t) \rightarrow -\frac{f_0}{\omega^2} e^{i\omega t} \quad (\text{small amplitude}).$$

Amplitude and peak.

The complex amplitude is

$$A(\omega) = \frac{f_0}{\omega_0^2 - \omega^2 + i\frac{\omega}{\tau}},$$

$$|A(\omega)|^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2}.$$

Maximal response occurs at

$$\frac{\partial}{\partial(\omega^2)} \left[(\omega_0^2 - \omega^2)^2 + \frac{\omega^2}{\tau^2} \right] = 0,$$

$$\omega_r = \sqrt{\omega_0^2 - \frac{1}{2\tau^2}}$$

$$\approx \omega_0 \left(1 - \frac{1}{4\omega_0^2\tau^2} \right).$$

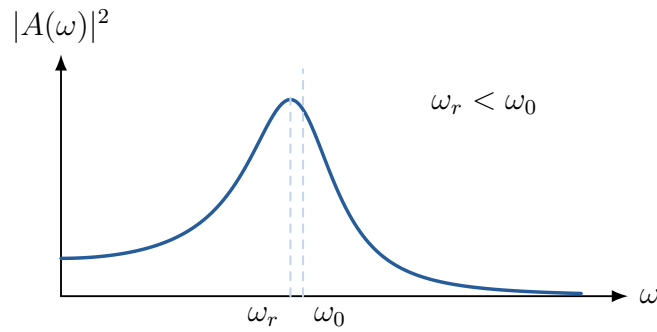


Fig. 21: Damping shifts the displacement-amplitude peak from ω_0 to ω_r .

Phase.

- For $\omega \ll \omega_0$: $\delta \rightarrow 0$ (in-phase; A is real).
- For $\omega \gg \omega_0$: $\delta \rightarrow \pi$ (out of phase by π ; A is real).

The phase angle follows from the real and imaginary parts of the response denominator.

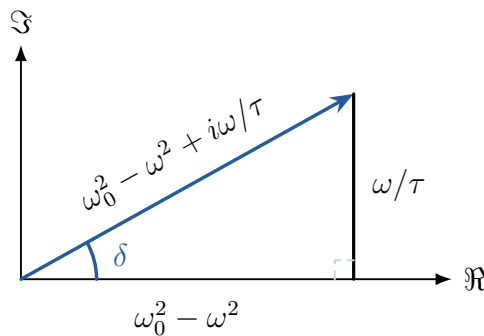


Fig. 22: The phase triangle relates δ to $\omega_0^2 - \omega^2$ and ω/τ .

Dissipation produces a frequency-dependent phase delay.

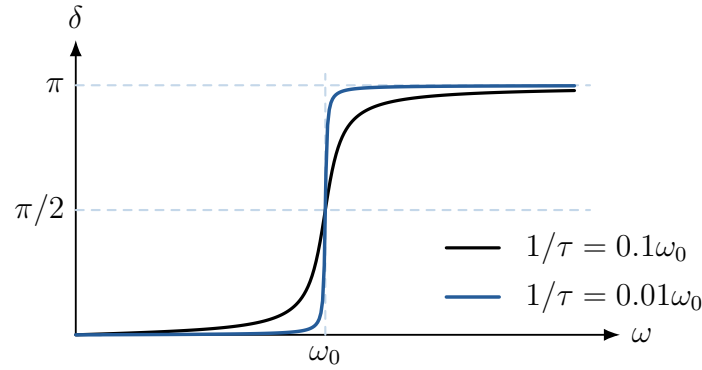


Fig. 23: The phase lag grows from 0 to π , passing through $\pi/2$ at ω_0 .

Width of the resonance (weak damping $1/\tau \ll \omega_0$).

Near $\omega \approx \omega_0$, write $\omega = \omega_0 + \Delta$ so that $\omega_0^2 - \omega^2 \approx -2\omega_0\Delta$. The half-maximum condition $|A(\omega)|^2 = \frac{1}{2} |A(\omega_0)|^2$ gives

$$4\omega_0^2\Delta^2 = \left(\frac{\omega_0}{\tau}\right)^2 \Rightarrow \Delta = \pm \frac{1}{2\tau}.$$

Thus the full width at half maximum is

$$\Gamma \equiv \omega_+ - \omega_- \approx \frac{1}{\tau}, \quad \omega_{\pm} \approx \omega_0 \pm \frac{1}{2\tau}.$$

Feynman perspective: Quality factor and what resonance measures.

For a weakly damped oscillator, the sharpness of resonance is summarized by

$$Q \equiv \frac{\omega_0}{\Gamma} \approx \omega_0\tau.$$

Large Q means that the system stores energy for many oscillations before dissipation removes it. Equivalently, resonance is not just a large amplitude; it is a balance between energy supplied by the drive and energy lost to damping. The transient part remembers the initial conditions, but the long-time resonant response measures the system itself.

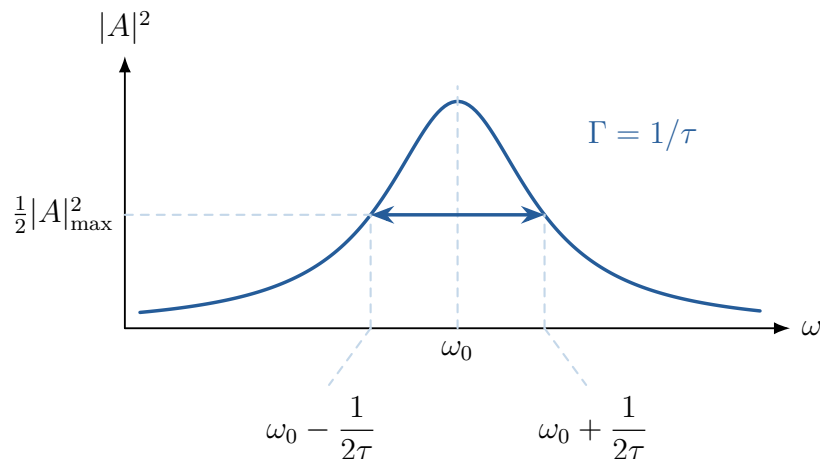


Fig. 24: The resonance width is measured between the half-maximum points of $|A|^2$.

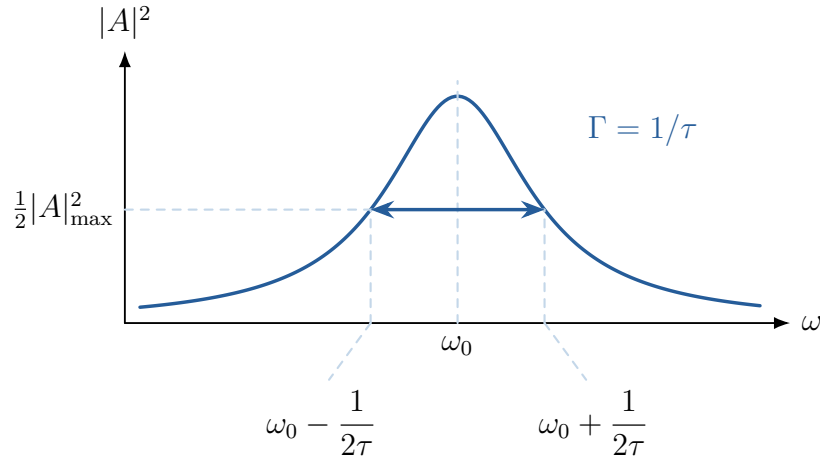


Fig. 25: Resonance curves illustrate frequency selectivity.

5.6.1 Radio tuning

Radio antennas are driven by multiple oscillating EM fields, e.g. F_a at ω_a and F_b at ω_b . By superposition, the circuit receives $x_a + x_b$. Yet a resonant LC circuit responds sharply near its natural frequency ω_0 . If tuned to ω_a , then $x_a \gg x_b$, so the output is dominated by station a . Changing C (or L) shifts $\omega_0 = 1/\sqrt{LC}$, allowing one to silence both or tune to ω_b . Thus, radio tuning illustrates superposition filtered by resonance.

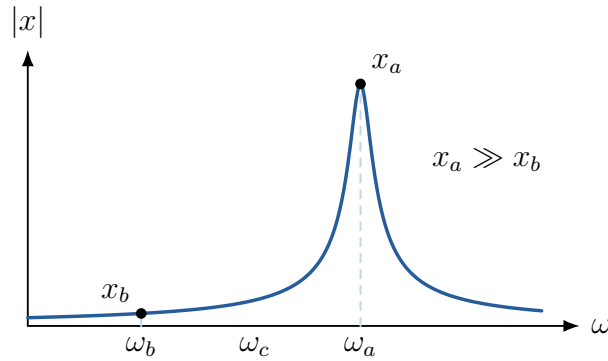


Fig. 26: Radio tuning selects one station using a narrow resonance curve.^[2]

5.6.2 Tuning in to dark matter

Could a radio detect dark matter? If dark matter consists of sufficiently light axions, it can behave as an oscillating field. In a static magnetic field, its coupling to electromagnetism could produce a tiny alternating signal. DMRadio searches for this signal using a tunable LC resonator and sensitive superconducting readout.^[3] The principle is familiar:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \quad \text{is tuned toward} \quad f_{\text{DM}} \simeq \frac{mc^2}{h}.$$

Changing the capacitance scans possible dark-matter masses. A high quality factor Q gives a narrow resonance, while cryogenic operation (低温运行) helps reduce thermal noise and losses.

The measured peak in Fig. 28 illustrates the resonator's response to thermal noise; it is not a dark-matter signal.^[4] A search looks for an additional narrow-band excess above the expected noise. The challenge connects familiar oscillation physics to precision measurement: enhance a weak response, understand the noise, and scan an unknown frequency.

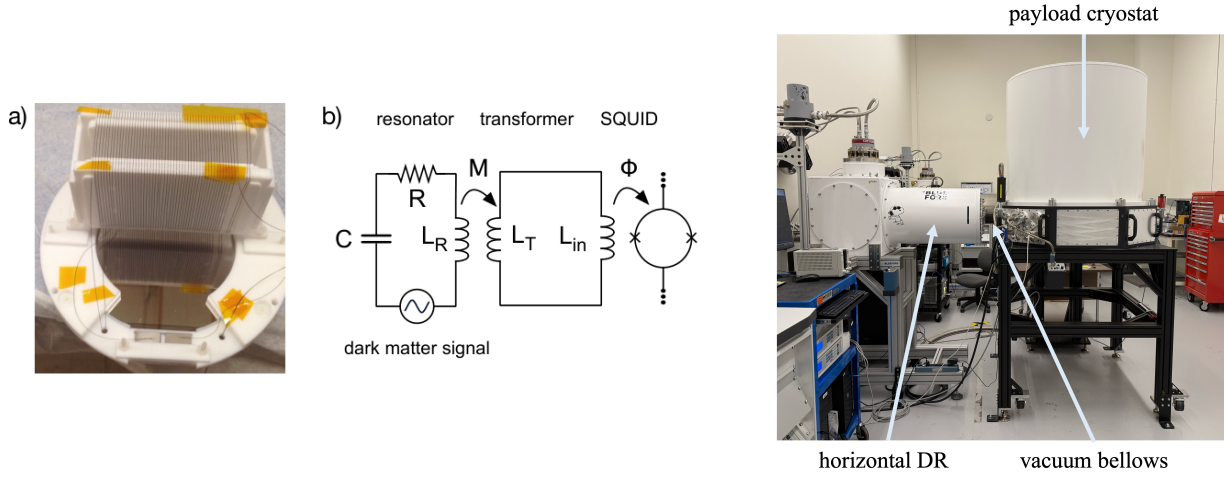


Fig. 27: From circuit to experiment. Left: an early DMRadio prototype and its equivalent circuit. Right: the DMRadio-50L cryogenic system. These show different stages of the experimental program.^[4, 5]

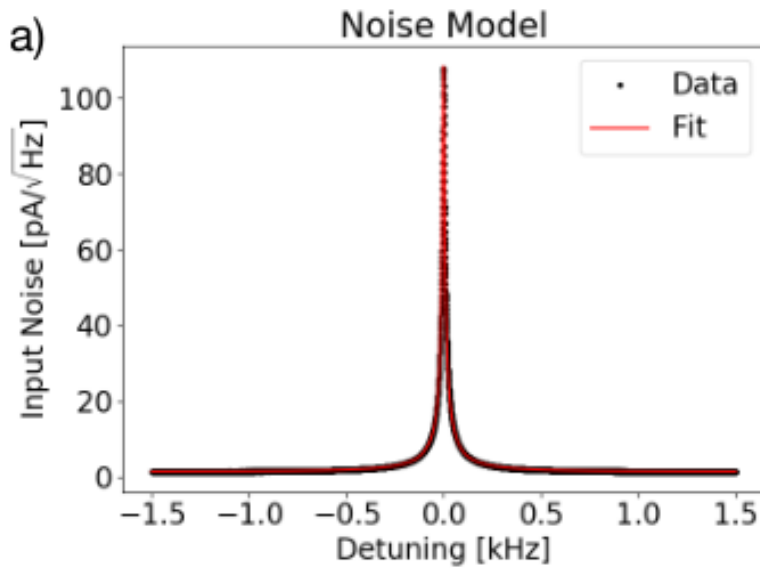


Fig. 28: Measured thermal-noise resonance of the early prototype near 492 kHz, with its fitted response. The power-spectrum linewidth determines $Q \simeq f_0/\Delta f$; the plotted vertical axis is amplitude spectral density.^[4]

5.7 Linear systems and superposition*

* *Optional enrichment* (拓展阅读, 不作要求) .

Suppose a system is driven by F_a (e.g. oscillatory with $\omega = \omega_a$), producing response x_a .

For another force F_b , the response is x_b . If both act together, the solution is simply

$$L(x_a + x_b) = L(x_a) + L(x_b) = F_a(t) + F_b(t),$$

so the total motion is $x_a + x_b$. This is the **principle of superposition**: a complicated force may be decomposed into simpler pieces, solved individually, and recombined.

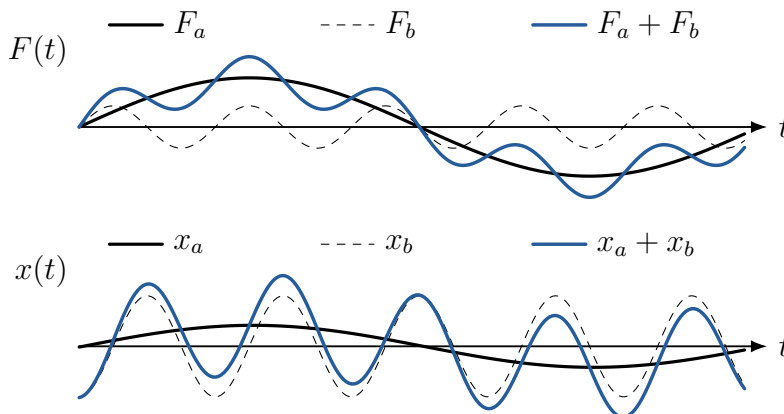


Fig. 29: Superposition of responses in a linear system.^[2]

An analogous result appears in electrostatics. A charge distribution q_a gives field $\mathbf{E}_a$, while q_b produces $\mathbf{E}_b$. Together,

$$\mathbf{E} = \mathbf{E}_a + \mathbf{E}_b,$$

so the total field is the vector sum. This works because Maxwell's equations are linear.

In summary, linear systems are powerful because they allow complex problems to be solved piece by piece and then assembled into the full solution.

5.7.1 Neutrino oscillations

Superposition also connects oscillations to quantum physics. Two sinusoidal waves with nearby frequencies produce beats (拍频) because their relative phase changes with time. A related interference principle underlies neutrino oscillations (中微子振荡).

Neutrinos are produced and detected in three flavors (味): electron, muon, and tau. A neutrino produced in one flavor is a quantum superposition of different mass states (质量态). As it propagates, these components accumulate different phases. Their interference changes the probability of detecting each flavor, depending on the travel distance and neutrino energy.^[6]

Here the oscillating quantity is a flavor-detection probability. Classical beats provide an analogy for the changing relative phase; a full description requires quantum mechanics. The connection is the power of superposition: simple component waves can combine to produce a changing observable.

Oscillators show how Newton's law becomes solvable through linearity. Next, central-force motion is solvable for a different reason: geometry and conserved angular momentum.

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map follows this chapter; the final step applies these tools to the oscillator equation.
https://www.feynmanlectures.caltech.edu/I_22.html

- [2] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. I, Ch. 25, *Linear Systems and Review*. Radio-tuning and superposition diagrams follow Figs. 25–3 and 25–1. https://www.feynmanlectures.caltech.edu/I_25.html
- [3] DMRadio Collaboration, *Projected sensitivity of DMRadio- m^3 : A search for the QCD axion below $1\,\mu\text{eV}$* , Phys. Rev. D 106, 103008 (2022). <https://arxiv.org/abs/2204.13781>
- [4] A. Phipps et al., *Exclusion limits on hidden-photon dark matter near 2 neV from a fixed-frequency superconducting lumped-element resonator* (2019), Figs. 1 and 2(a). <https://arxiv.org/abs/1906.08814>
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