

Lecture 4: Newton's laws of motion(I)

From description to cause. Galileo describes how bodies move; Newton asks what changes their motion. The shift from kinematics to dynamics is the first major step in Newton's program.

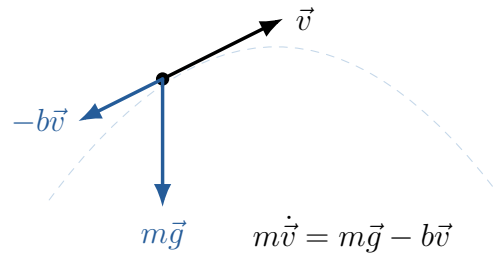


Fig. 1

Outline:

- Newton's standing on Galileo's shoulders
- Law of inertia
- Newton's 2nd law-concepts of force, mass
- Projectile motion with resistance
- linear vs. quadratic drags

$$\begin{cases} m\dot{v}_x = -bv_x, \\ m\dot{v}_y = -mg - bv_y. \end{cases} \quad \begin{aligned} \tau &= \frac{m}{b}, \\ v_\infty &= \frac{mg}{b}. \end{aligned}$$

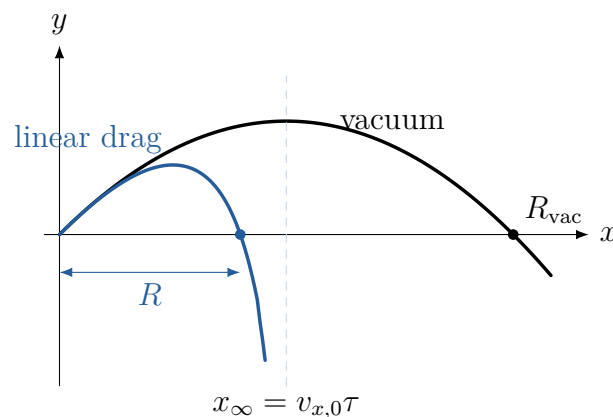


Fig. 2

$$R \approx R_{\text{vac}} \left[1 - \frac{4}{3} \frac{v_{y0}}{v_{(\infty)}} \right]$$

4.1 From Galileo to Newton

4.1.1 Inertia and inertial frames

Galileo: discovery of the principle of inertia (Newton's first law of motion). If a body is not exerted a force on it, it keeps still, or it maintains the motion at a constant velocity.

Inertial frames are frames in which an isolated body moves at constant velocity. A freely moving object can appear to accelerate relative to an accelerating train. Identifying inertial frames and identifying negligible interactions must be tested together across many bodies and experiments. The first law asserts the existence of such frames within the Newtonian approximation.

See the discussion of inertia and the role of friction in Feynman, Ch. 9.^[1]

4.1.2 Force, mass, and the second law

Standing on the shoulders of Galileo, Newton improved the concept of force to a much more general and abstract level. Using his imagination, Newton generalized the gravity for falling apples to the motions of moon, planets, etc! Newton also proposed the concept of “mass” as a measurement of inertia. Then we can compare the accelerations among different objects and different forces!

See the distinction between weight and inertia in Feynman, Ch. 9.^[1]

Newton's 2nd law: Time-rate-of-change of momentum is proportional to force

$$\vec{F} = \frac{d}{dt}(m\vec{v}).$$

For a fixed-mass body in non-relativistic mechanics, $v \ll c$ and $\vec{p} = m\vec{v}$. In this case, Newton's law becomes

$$\vec{F} = m \frac{d\vec{v}}{dt} = m \frac{d^2\vec{r}}{dt^2} = m\vec{a}$$

This is a 2nd order differential equation, and also a vector equation. Acceleration not only could mean the change of speed, but also could mean the change of motion direction!

4.2 Projectile motion with resistance

4.2.1 Linear and quadratic drag

projectile (抛射体); *drag* (阻力); *viscosity* (黏度); *terminal speed* (终端速率)。If we have the knowledge of force, in principle, we should be able to solve the equation of motion. Now we can consider a more realistic projectile motion by considering the air resistance $\vec{f} = -f(v)\hat{v}$, which is always in the opposite direction of velocity causing dissipation. (The dissipation power $\vec{f} \cdot \vec{v} < 0$).

For a sphere at low Reynolds number, $\text{Re} = \rho_a v D / \eta \ll 1$, Stokes' law gives a drag proportional to speed:

$$f_{\text{lin}} = 3\pi\eta Dv,$$

where η is the viscosity of the fluid, D is the droplet diameter. The unit of η is $\text{kg}/(\text{m} \cdot \text{s})$. Actually, the resistance also contains the quadratic part. In a short time Δt , the projectile travels at a distance $v\Delta t$. It pushes the air of the volume $vA\Delta t$ to the velocity v .

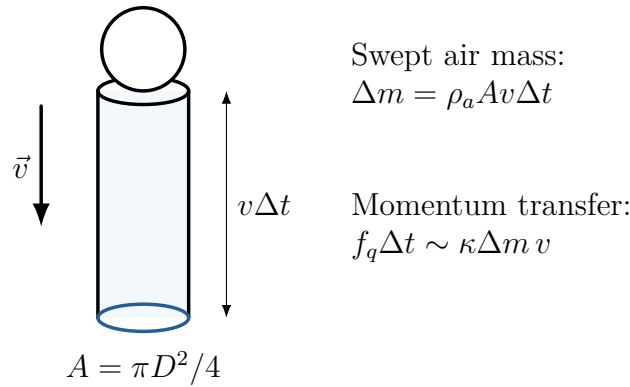


Fig. 3

Here κ is a dimensionless coefficient of order unity for this momentum-transfer estimate.

$$f_q \cdot \Delta t = \kappa \rho_{\text{air}} v A \Delta t \cdot v \Rightarrow f_q = \frac{\kappa \pi}{4} D^2 \rho_a v^2$$

4.2.2 Comparing the drag models

$$\begin{aligned} \frac{f_q}{f_{\text{lin}}} &= \frac{\kappa \pi D^2 \rho_a v^2}{4 \cdot 3\pi \eta D v} \\ &= \frac{\kappa}{12} \underbrace{\frac{D v \rho_a}{\eta}}_{\text{Reynolds number}}. \end{aligned}$$

plug in $\eta = 1.7 \times 10^{-5} \text{ kg}/(\text{m} \cdot \text{s})$, $\rho_a = 1.29 \text{ kg}/\text{m}^3$

$$\frac{f_q}{f_{\text{lin}}} = \left[\frac{6 \times 10^3 \text{ s}}{\text{m}^2} \right] \kappa D v$$

$$\longrightarrow 1.5 \times 10^3 \left(\frac{\text{s}}{\text{m}^2} \right) D v.$$

-base ball $D = 7 \text{ cm}$ and $v = 5 \text{ m/s} \Rightarrow f_q/f_{\text{lin}} = 600$

-rain droplet $D = 1 \text{ mm}$, $v = 0.6 \text{ m/s}$ $f_q/f_{\text{lin}} \simeq 1$

-Milikan oil drop $D = 1.5 \mu\text{m}$, $v = 5 \times 10^{-5} \text{ m/s}$ $f_q/f_{\text{lin}} \simeq 10^{-7}$.

For big and fast projectiles, the quadratic drag is more important, while for small and slow projectiles, the linear one is more important.

4.2.3 Drag on an airplane

What force does the air exert on a flying airplane? Air flows around the fuselage (机身) and wings, and swirls in the wake (尾流). Despite this complexity, wind-tunnel measurements often give a useful approximation: the drag is proportional to the square of the speed, $F_{\text{drag}} \approx C v^2$, over a suitable range of conditions.

Feynman emphasizes the status of this relation: it summarizes many complicated processes. More precise measurements reveal further dependence on speed, shape, and flow conditions. At sufficiently slow motion in a viscous fluid, drag is instead approximately proportional to speed. Each simple force model has a domain of validity.

Even a wing tested alone can experience a different force from the same wing attached to an airplane: the rest of the airplane changes the air reaching it. The coefficient in a drag law contains information about the whole flow. This motivates examining both linear and quadratic drag before predicting a projectile's motion.^[2]

Physical understanding and practical use of fluid equations long preceded the recently announced mathematical proof about their possible singular behavior.¹

4.3 Motion with linear air resistance

$$m\ddot{\vec{r}} = m\dot{\vec{v}} = m\vec{g} - b\vec{v}.$$

$$\begin{cases} m\dot{v}_x = -bv_x, \\ m\dot{v}_y = -mg - bv_y. \end{cases}$$

where $b=3\pi\eta D$. Note Newton's notation here.

¹The Navier–Stokes equations (纳维–斯托克斯方程) describe a continuous fluid, averaging over individual molecules, and can generate complex vortices and turbulence (湍流).^[3] On September 8, 2026, OpenAI announced a proof of finite-time singularity formation in the three-dimensional incompressible equations under a smooth external force.^[4]

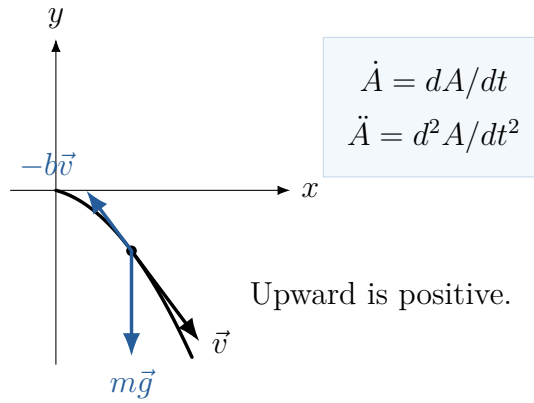


Fig. 4

4.3.1 Horizontal motion

See the discussion of the meaning and numerical solution of dynamical equations in Feynman, Ch. 9.^[5]

$$\frac{dv_x}{v_x} = -\frac{b}{m} dt,$$

$$v_x(t) = Ae^{-t/\tau}, \quad \tau = \frac{m}{b}.$$

Here τ is the time constant. Combine with $v_x(t=0) = v_{x,0} \Rightarrow A = v_{x,0}$.

$$v_x = \frac{dx}{dt},$$

$$\begin{aligned} x(t) &= x(0) + \int_0^t v_x(t') dt' \\ &= x(0) + v_{x,0} \int_0^t e^{-t'/\tau} dt'. \end{aligned}$$

Set $x(0) = 0 \Rightarrow x(t) = x_\infty (1 - e^{-t/\tau})$, where $x_\infty = v_{x,0}\tau$.

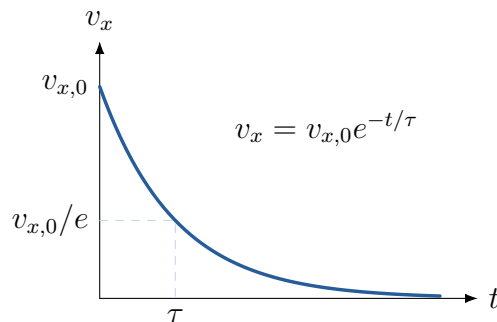


Fig. 5

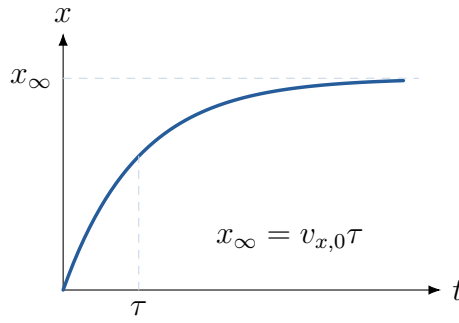


Fig. 6

4.3.2 Vertical motion and terminal speed

Along the y -direction, the inhomogeneous first-order ODE and its steady-state limit are

$$\dot{v}_y = -g - \frac{b}{m}v_y,$$

$$\dot{v}_y \longrightarrow 0 \quad \Rightarrow \quad v_y(t \longrightarrow \infty) = -\frac{mg}{b} \equiv -v(\infty).$$

$$v_y(t) = \underbrace{Ae^{-t/\tau}}_{\text{homogeneous solution}} \underbrace{-v(\infty)}_{\text{particular solution}}$$

plug in $v_y(t=0) = v_{y,0} \Rightarrow A = v_{y,0} + v(\infty)$

$$\Rightarrow v_y(t) = v_{y,0}e^{-t/\tau} - v(\infty)(1 - e^{-t/\tau})$$

With $y(0) = 0$,

$$\begin{aligned} y(t) &= \int_0^t v_y(t') dt' \\ &= -v(\infty)t \\ &\quad + (v_{y,0} + v(\infty))\tau(1 - e^{-t/\tau}). \end{aligned}$$

If $v_y(t=0) = 0$, we have $v_y = -v(\infty)(1 - e^{-t/\tau})$

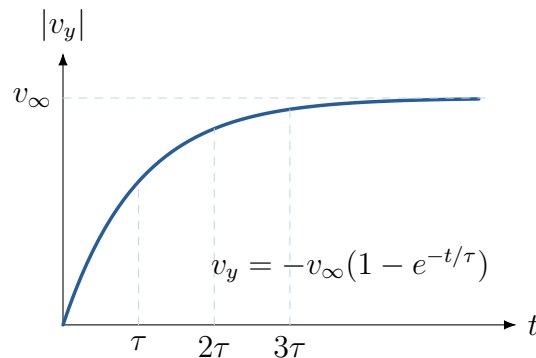


Fig. 7

time	percent of $v(\infty)$
τ	63%
2τ	86%
3τ	95%

4.3.3 Orders of magnitude

(1) Terminal speed:

$$v(\infty) = \frac{mg}{b}$$

$$= \frac{\rho\pi D^3 g}{6 \cdot 3\pi\eta D} = \frac{\rho D^2 g}{18\eta}.$$

For an oil drop in the Millikan experiment,

$$D = 1.5\mu\text{m}. \quad \rho = 840 \text{ kg/m}^3 \Rightarrow v(\infty) = 6.1 \times 10^{-5} \text{ m/s}.$$

But for size $D = 0.2 \text{ mm} \Rightarrow v(\infty) = 1.3 \text{ m/s}$ of drizzle drop

(2) Time scale:

$$\tau = \frac{m}{b} = \frac{v(\infty)}{g}.$$

For millikan drop $\Rightarrow \tau \approx 6 \times 10^{-6} \text{ s}$

For drizzle drop $\rightarrow \tau \approx 0.13 \text{ s}$.

4.4 Trajectory and range

4.4.1 Trajectory

With $x(0) = y(0) = 0$,

$$x(t) = v_{x0}\tau (1 - e^{-t/\tau}),$$

$$y(t) = (v_{y0} + v(\infty))\tau (1 - e^{-t/\tau}) - v(\infty)t.$$

From the horizontal displacement,

$$t = -\tau \ln \left(1 - \frac{x}{v_{x0}\tau} \right).$$

$$y = \frac{v_{y0} + v_{\infty}}{v_{x0}}x + v(\infty)\tau \ln \left(1 - \frac{x}{v_{x0}\tau} \right)$$

x cannot exceed $x_{\infty} = v_{x0}\tau$.

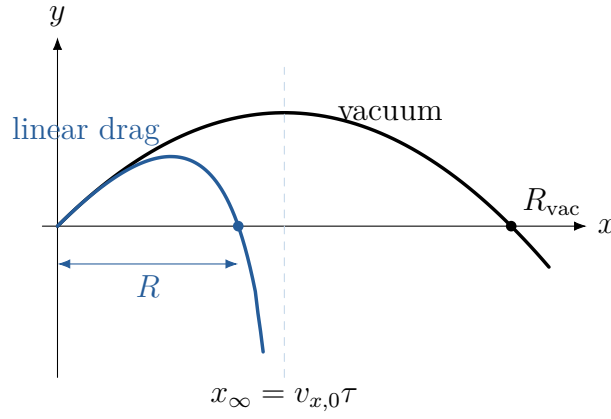


Fig. 8

4.4.2 Range in the weak-drag limit

The horizontal range in vacuum

$$R_{vac} = \frac{2v_{x0}v_{y0}}{g}.$$

Now with resistance, we solve

$$\frac{v_{y0} + v(\infty)}{v_{x0}} R + v(\infty)\tau \ln \left(1 - \frac{R}{v_{x0}\tau} \right) = 0.$$

In the limit of small resistance, τ is large, that $\frac{R}{v_{x0}\tau} \ll 1$ we use

$$\ln(1 - \epsilon) = - \left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \dots \right).$$

$$\begin{aligned} \frac{v_{y0} + v(\infty)}{v_{x0}} R &= v(\infty)\tau \left[\frac{R}{v_{x0}\tau} + \frac{1}{2} \left(\frac{R}{v_{x0}\tau} \right)^2 \right. \\ &\quad \left. + \frac{1}{3} \left(\frac{R}{v_{x0}\tau} \right)^3 \right], \end{aligned}$$

$$\frac{v_{y0}}{v_{x0}} \frac{R}{v(\infty)\tau} = \frac{1}{2} \left(\frac{R}{v_{x0}\tau} \right)^2 + \frac{1}{3} \left(\frac{R}{v_{x0}\tau} \right)^3,$$

$$R = \frac{2v_{x0}v_{y0}}{g} - \frac{2R^2}{3v_{x0}\tau}.$$

$$R \approx R_{vac} - \frac{2}{3v_{x0}\tau} \frac{4v_{x0}^2 v_{y0}^2}{g^2}$$

$$= R_{vac} \left[1 - \frac{4}{3} \frac{v_{y0}}{v(\infty)} \right].$$

when $\frac{v_{y0}}{v(\infty)} \simeq 1$, the effect of air-resistance **cannot be neglected!**

4.4.3 Numerical estimates

A metal pellets $D = 0.2$ mm, $v = 1$ m/s at angle 45° .

The range in the absence of resistance:

$$\begin{aligned} R_{\text{vac}} &= \frac{2v_{x0}v_{y0}}{g} = \frac{v^2 \sin 2\theta}{g} \\ &= \frac{1}{9.8} \text{ m} \approx 10.2 \text{ cm}. \end{aligned}$$

For a dense metal, use

$$v(\infty) = \frac{\rho D^2 g}{18\eta}, \quad \rho = 16 \text{ g/cm}^3.$$

$$D = 0.2 \text{ mm}.$$

$$\eta = 1.7 \times 10^{-5} \text{ kg/m} \cdot \text{s}$$

Thus $v(\infty) = 21$ m/s, and the correction is

$$\frac{4}{3} \frac{1 \times 0.7}{21} \simeq 5\%.$$

For Al, $\rho = 2.7 \text{ g/cm}^3$, which is approximately $1/6$ of the assumed dense-metal value. Hence, the correction is 6 times larger, $\simeq 30\%$.

References

- [1] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. I, [Sec. 9–1, Momentum and force](#). Inertia, friction, and the distinction between weight and mass.
- [2] Feynman, Leighton, and Sands, Vol. I, [Sec. 12–2, “Friction.”](#) Airplane and isolated-wing forces; linear and quadratic drag. The model-testing discussion extends these examples.
- [3] NASA Glenn Research Center, [Navier–Stokes Equation](#). Continuum fluid dynamics and numerical modeling.
- [4] OpenAI, September 8, 2026, [On the Navier–Stokes Millennium Prize Problem](#). Announced finite-time singularity result with smooth forcing; independent verification pending at the September 9, 2026 lecture update.
- [5] Feynman, Leighton, and Sands, Vol. I, [Sec. 9–5, Meaning of the dynamical equations](#), and [Sec. 9–6, Numerical solution of the equations](#).