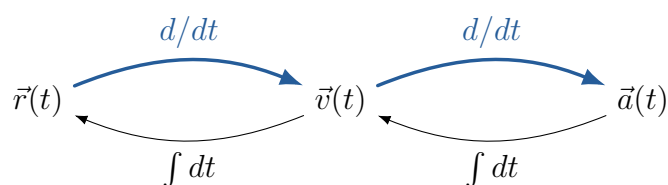


Lecture 3: Motion: Zeno's paradox, displacement, velocity, acceleration

From paradox to measurement. Zeno makes motion look logically fragile; Galileo makes it measurable. This lecture follows that turn: motion becomes a function of time, and the derivative turns a philosophical puzzle into a physical observable.



Integration also requires initial conditions.

Motion is a continuous process while our logical reasoning is discrete, or, step by step. How to use discrete steps of reasoning to precisely describe a continuous motion is a highly non-trivial problem. The ancient Greeks had already paid attention to this problem as represented by the Zeno paradox. In fact, in order for a deep understanding, an infinitesimal analysis is necessary, which are the watershed ridge between the advanced mathematics and elementary one.

3.1 Zeno's paradox

paradox (悖论): 看似合理的推理导出与直觉或事实相冲突的结论。Zeno of Elea (490-430BC) was a Greek philosopher. He raised a paradox that Achilles, a hero of the Trojan War in Greek mythology, could not catch up with a tortoise. Aristotle recounts the argument that the pursuer must first reach the point from which the pursued started, while the latter has already moved ahead.

To be concrete, assume that Achilles' velocity is $v_a = 10$ m/s and the tortoise's $v_t = 0.1$ m/s. Initially, Achilles is located at $x_0 = 0$ m and the tortoise is ahead of Achilles at $x_1 = 99$ m, and then the distance between them $s_1 = x_1 - x_0 = 99$ m. This is a math problem that we learned how to solve in elementary school. Assume that it takes Achilles the time T to overtake the tortoise, then it can easily derived that,

$$T = \frac{s_1}{v_a - v_t} = \frac{99}{9.9} s = 10s. \quad (3.1)$$

Nevertheless, Zeno provided a different perspective. He divided this chasing process into a series of steps: During step 1, Achilles reaches the initial position of the tortoise x_1 . Meanwhile the tortoise moved ahead to x_2 . During step 2, Achilles reaches Δt_2 and the tortoise moved to x_3 , and so on. Since these steps can be repeated forever, Zeno concluded that Achilles could never overtake the tortoise.

Certainly, this conclusion should not make sense. Where is the flaw in Zeno's reasoning? Let us denote the time interval spent during the n -the step as Δt_n , and then the total time spent should be,

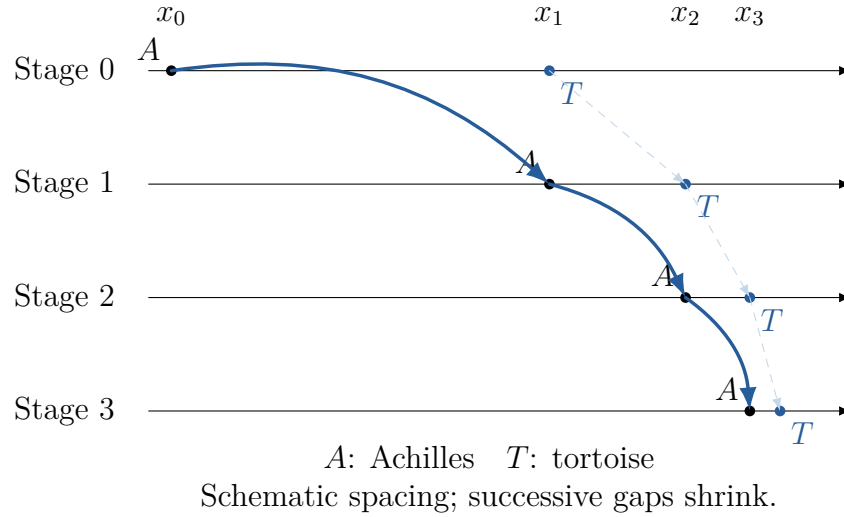


Figure 3.1 Zeno's paradox that Achilles cannot overtake a tortoise. Achilles' and the tortoise's initial positions are denoted x_0 and x_1 , respectively. When Achilles arrives at x_1 , the tortoise moves ahead to x_2 . Then Achilles arrives at x_2 , and the tortoise moves to x_3 , and so on. This paradox shows the gap between our perception and the outside world in that our thinking is step by step while the motion is continuous.

$$T = \Delta t_1 + \Delta t_2 + \Delta t_3 + \dots, \quad (3.2)$$

The question is that even though there is an infinite number of terms in this summation, does it really mean that the sum is infinite, or, could it still be finite?

Feynman perspective: Infinitely many steps.

Dividing the last metre repeatedly in half gives infinitely many pieces. Can their total length remain finite? Apply the same question to Achilles' time intervals. The number of stages alone does not determine the total duration. *Discuss: which assumption in Zeno's argument needs examination?* ^[1]

To see what really happens, we need to analyze more carefully each step. During step one, the time spent is

$$\Delta t_1 = \frac{s_1}{v_a} = 9.9 \text{ s}. \quad (3.3)$$

Meanwhile the distance that the tortoise moved is

$$s_2 = x_2 - x_1 = v_t \Delta t_1 = s_1 \frac{v_t}{v_a} = 0.99 \text{ m} \quad (3.4)$$

Then the time interval Δt_2 spent during step 2 is

$$\Delta t_2 = \frac{s_2}{v_a} = \Delta t_1 \frac{v_t}{v_a} = 0.099 \text{ s}, \quad (3.5)$$

and the distance during step two that the tortoise moved is

$$s_3 = x_3 - x_2 = v_t \Delta t_2 = 0.0099 \text{ m.} \quad (3.6)$$

By a similar reasoning, during the n -th step, Achilles takes the time Δt_n as

$$\Delta t_n = \Delta t_1 q^{n-1} \quad (3.7)$$

where $q = v_t/v_a$.

In elementary mathematics, we only learned how to sum finite terms. For example, we define

$$\begin{aligned} T_n &= \Delta t_1 (1 + q + q^2 + \dots + q^{n-1}) \\ &= \Delta t_1 \frac{1 - q^n}{1 - q} \\ &= 10 \times (1 - (0.01)^n) \text{ s.} \end{aligned} \quad (3.8)$$

Hence, we arrive at

$$\begin{aligned} T_1 &= \Delta t_1 = 9.9 \text{ s} \\ T_2 &= \Delta t_1 + \Delta t_2 = 9.999 \text{ s} \\ &\dots \\ T_n &= \Delta t_1 + \Delta t_2 + \dots + \Delta t_n = 9.99 \dots 99 \text{ s} \end{aligned} \quad (3.9)$$

So far everything is elementary mathematics.

The breakthrough actually arises when $n \rightarrow \infty$ is taken. In this case, literally we have

$$T = \Delta t_1 + \Delta t_2 + \dots \quad (3.10)$$

which gives rise to $T = 9.99 \dots \text{s}$. Since each term $\Delta t_n = \Delta t_1 q^{n-1}$, T is expressed order by order of q . Since $|q| < 1$, the higher order the term is, the smaller its contribution is. Hence, Eq. (3.10) is a perturbation theory. In contrast, Eq. (3.1) is a non-pertubative theory.

Compared to Eq. (3.1), the natural question is: Should 9.999.... be taken precisely as 10, or not? How to understand 9.99...? Let us check:

$$\begin{aligned} 10 - T_1 &= 10 - 9.9 = 0.1 \\ 10 - T_2 &= 10 - 9.999 = 0.001 \\ 10 - T_3 &= 10 - 9.99999 = 0.00001 \end{aligned} \quad (3.11)$$

Even exhausting our life, we could only perform the above process at finite steps. The great leap from elementary math to advanced one actually lies in that we are willing to

accept the difference between $9.999\dots$ and 10 is precisely 0, i.e., no approximation. This is because the difference between 10 and $9.99\dots$ could be as small as you would like at any precision. Give me a precision ϵ , say, 10^{-2n} , we have $|10 - T_m| < \epsilon$ as long as $m > n$. Formally, it is denoted as

$$\lim_{n \rightarrow \infty} T_n = \lim_{n \rightarrow \infty} 9.99\dots 9 = 10 \quad (3.12)$$

This should not be viewed as a derivation; it is better understood as a definition. Formally, mathematics supplies the completeness axiom: every monotone bounded sequence has a finite limit. Here we have

$$\begin{aligned} T &= \lim_{n \rightarrow \infty} T_n = \frac{\Delta t_1}{1 - q} \lim_{n \rightarrow \infty} (1 - q^n) = \frac{\Delta t_1}{1 - q} \\ &= \frac{s_1}{v_a (1 - v_s/v_a)} = \frac{s_1}{v_a - v_s} \end{aligned} \quad (3.13)$$

3.2 Analytic continuation — let divergent series make sense

analytic continuation (解析延拓); *convergent / divergent series* (收敛/发散级数); *perturbation theory* (微扰理论)。The convergence of the geometric series relies on the common ratio $|q| < 1$. If $|q| > 1$, then the series diverges. Nevertheless, for physicists, a divergent series still makes sense in many situations. The key is the interpretation.

If we switch the positions of Achilles and the tortoise, then $q = v_a/v_s > 1$ and the geometric series of Eq. (3.10) diverges. Since the tortoise's speed is smaller than Achilles, there is no way for it to overtake Achilles. However, if we literally take Eq. (3.13) to see what happens, it becomes

$$T = \frac{\Delta t_1}{1 - q} = \frac{s_1}{v_s (1 - v_a/v_s)} = -\frac{s_1}{v_a - v_s} \quad (3.14)$$

Although each term in the series is positive, we arrive at a negative T . It perfectly makes sense as long as we extrapolate the motions of Achilles and the tortoise from the past to the future, actually they meet before the time zero.

What does a negative catching time mean? Imagine running a recording of the motion backwards. If the two objects maintain their velocities, a negative T tells us that they met before our chosen starting time.

Zeno's step-by-step construction works when each successive time interval becomes smaller. When those intervals grow, adding them no longer gives a finite catching time. We can still use the equation of motion to find when the two trajectories meet.

This illustrates the idea behind **analytic continuation** (解析延拓): an expression obtained in one range can sometimes describe the same problem beyond that range. Its physical meaning must be checked—in this example, the solution describes a meeting in the past.

For comparison, write the time in units of the first interval, $f(q) = T/\Delta t_1$. Zeno's construction gives

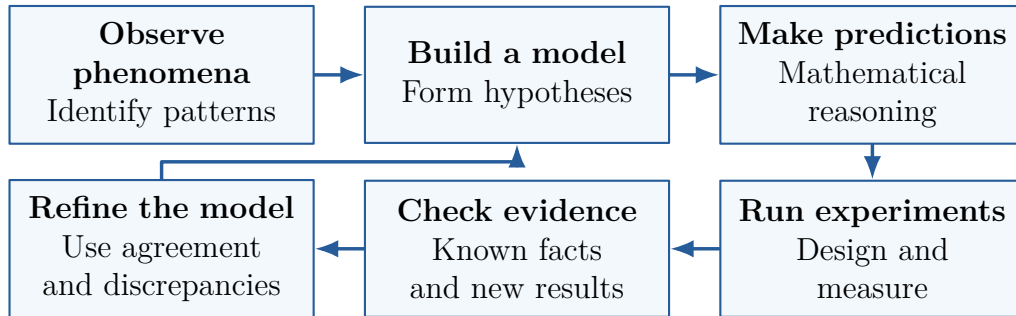
$$f(q) = \sum_{n=0}^{\infty} q^n. \quad (3.15)$$

For $|q| < 1$, this sum equals

$$f(q) = \frac{1}{1-q}. \quad (3.16)$$

For $q > 1$, Eq. (3.16) gives the negative meeting time discussed above. The series in Eq. (3.15) remains divergent there.

3.3 Galileo's study on the uniform acceleration



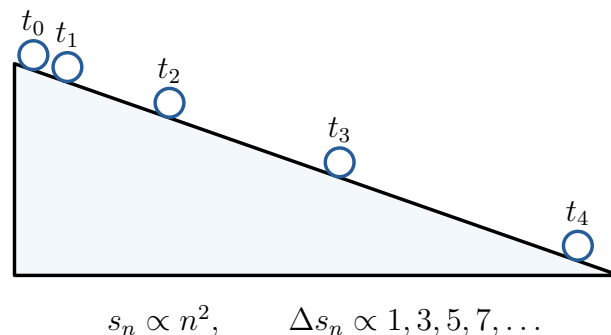
Galileo (伽利略·伽利莱, 1564–1642, 意大利物理学家、天文学家) helped establish a way of studying nature grounded in observation and experiment. Patterns in phenomena guide mathematical models and hypotheses, which lead to testable predictions. A theory is judged by how well it explains known facts and predicts new results.

The difficulties that Galileo faced were enormous:

1. Experiment difficulty — lack of precision measurements of time
2. Conceptual level — instant-velocity, acceleration, etc
3. Math level — pre-calculus time.

Laws are hidden in phenomena!

Free-fall motion is too fast to observe; hence, Galileo turned to study motion on a slope to slow down the motion such that the observation is easier.



Feynman perspective: Making motion measurable.

Imagine investigating free fall without an accurate stopwatch. An inclined track slows the motion enough to compare positions at equal times. Feynman describes pulse beats

as a possible timing reference. The cumulative distances follow $1, 4, 9, 16, \dots$; successive distances follow $1, 3, 5, 7, \dots$. *Discuss: how can changing the apparatus reveal a law?* ^[2]

3.3.1 Galileo's data

punti: 原始记录中的长度单位; *tick*: 计时的一拍。此处比较距离的比例, 无需换算成米。

t	1	2	3	4	5	6	7	8	tick (time)
S	33	130	298	526	824	1192	1620	2104	punti
ΔS divided by 33		97	168	228	298	368	428	484	
	1	2.9	5.1	6.9	9.0	11.2	13	14.6	$\propto t^2$

Considering possible experimental errors, Galileo observed the law that during an integral small time interval, the distance traveled forms an arithmetic progression — idealization.

$$1 : 3 : 5 : 7 : 9 : 11 : 13 : 15$$

Then the distance traveled from the starting time is

$$1, 1 + 3 = 4, 1 + 3 + 5 = 9, 1 + 3 + 5 + 7 = 16, \dots$$

$$\Rightarrow S(t) \propto t^2$$

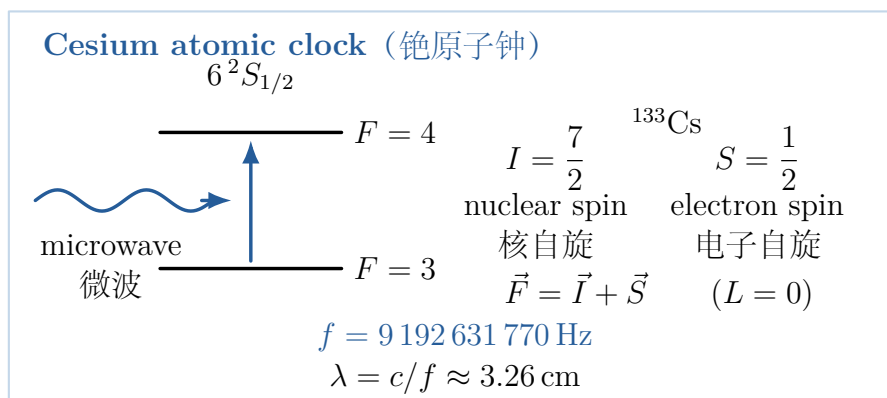
What's time? If no one asks me, I know what it is. If I wish to explain to him who asks, I don't know — **Saint Augustine** (圣奥古斯丁, 哲学家、神学家, 354–430).

Then how to measure time? Use naturally existing periods — year, month, day, etc. Or a better ones easier to control — the hour glass. But they cannot provide the enough precision even for a free fall motion!

Galileo expressed that water clocks were used to measure time in his experiments. But it's hard to imagine that the necessary precision could be achieved without prior knowledge of the motion. Physics history experts suggested that he might clap to the beat to divide time into short equal intervals. He was a musician!

Time-unit: Comité international des poids et mesures (CIPM) 2018: The second is defined by taking the fixed numerical value of the C_s frequency, ΔV_{CS} , the unperturbed ground state hyperfine transition frequency of the ^{133}Cs atom, to be 9,192,631,770 when expressed in the unit Hz , which is equal to s^{-1} .

(temperature $T = 0 \text{ K}$, average sea level, at rest, zero magnetic field).



Question: is this the most precise time **in 2026**?

Keyword cue: nuclear clock (核钟) . ^{229}Th (钍-229) ; nuclear transition (核能级跃迁) ; vacuum-ultraviolet laser (真空紫外激光) ; reduced environmental sensitivity (更低的环境敏感性) .^[4]

Feynman perspective: Who checks the clock?.

We gain confidence in time measurements through agreement between independent physical processes. Feynman compares the day with an hourglass, then uses pendulums and faster oscillators to measure shorter intervals. Agreement supports a consistent measure of time. If two clocks disagree, further comparisons and checks of their physical conditions help identify the cause.^[3]

Discussion: do accurate clocks always give the same reading? Two atomic clocks are initially synchronized. One is then placed higher than the other near Earth's surface. Will they continue to agree? Could both clocks work correctly and still accumulate different elapsed times?^[5]

3.4 Conceptual difficulty — instantaneous velocity

Feynman perspective: Only seven minutes.

Feynman tells a joke about a driver accused of travelling at sixty miles per hour: the driver protests that the journey lasted only seven minutes. “Miles per hour” describes a rate: at a constant sixty miles per hour, the car would cover seven miles in seven minutes. Even while the car slows down, we can describe its speed at a particular instant by measuring its motion over shorter and shorter time intervals. ^[1]

At Galileo's time, people had difficulty to define instantaneous velocity clearly. For example, consider the reverse motion of free fall — the vertical projectile motion. We know that the motion is slowed down as the projectile reaches the peak.

Let us divide the motion into n parts equally, and each step is h/n . As the projectile moves upward, finally $v \rightarrow 0$. It means it will take projectile more and more time to complete each step. Since we can divide the process into infinitely many steps, it's not clear whether the projectile could complete the motion in a finite period of time.

Or if we reverse the motion and consider the free fall, it means that if the falling object starts from velocity zero, how could it start to fall at all!

Of course, we do know from our daily observation, both the projectile and the falling object will complete the motion. The ultimate solution is actually not simple, which rely on our concept of infinitesimal quantities — the sum of infinite number of time intervals can still converge to a finite amount of time period — pretty much similar to the Zeno's paradox.

Galileo's response is that an object passes a point instantaneously. It occurs at a particular time "t", but it does not take an interval of time. Using modern language, it means

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{S(t + \Delta t) - S(t)}{\Delta t} = \frac{dS(t)}{dt}$$

or $\Delta t = \frac{\Delta S}{v(t)} \rightarrow 0$ as $\Delta S \rightarrow 0$.

3.4.1 Series expansion — frequently used formula

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots \end{aligned}$$

3.4.2 Example of derivative

$$\begin{aligned} \frac{d}{dx}e^x &= \lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} \\ &= e^x \lim_{\Delta x \rightarrow 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x, \end{aligned}$$

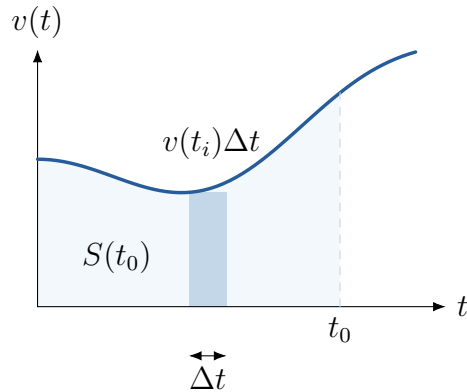
$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = (\ln a)e^{x \ln a} = (\ln a)a^x,$$

$$\begin{aligned} \frac{d}{dx}x^n &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\ &= x^n \lim_{\Delta x \rightarrow 0} \frac{(1 + \Delta x/x)^n - 1}{\Delta x} \\ &= nx^{n-1}, \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\ln x &= \lim_{\Delta x \rightarrow 0} \frac{\ln(x + \Delta x) - \ln x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\ln(1 + \Delta x/x)}{\Delta x} = \frac{1}{x}. \end{aligned}$$

3.4.3 Velocity v.s. distance

Distance (displacement)



Set $S(t = 0) = 0$.

$$S(t_0) = \sum_i v(t_i) \Delta t \Rightarrow \int_0^{t_0} dt v(t) = S(t_0)$$

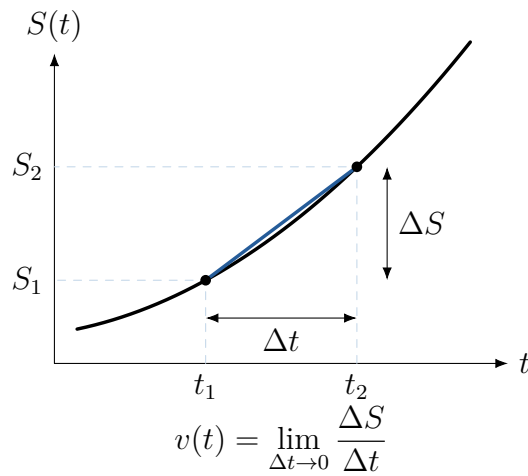
$$S(t_0 + \Delta t) - S(t_0) = \Delta t V(t_0)$$

$$\Rightarrow \lim_{\Delta t} \frac{S(t_0 + \Delta t) - S(t_0)}{\Delta t} = V(t_0)$$

$$\frac{dS(t)}{dt} = V(t)$$

Newton-Leibnitz formula:

$$\int_a^b f'(x) dx = f(b) - f(a)$$



$$t_2 - t_1 = \sum_i \frac{\Delta S(t_i)}{v} = \int_{s_1}^{s_2} \frac{dS}{v}.$$

3.5 Acceleration

$$\begin{aligned}
 a(t) &= \frac{dv(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} \\
 &= \frac{d^2 S(t)}{dt^2} \\
 &= \lim_{\Delta t \rightarrow 0} \frac{S(t + \Delta t) - 2S(t) + S(t - \Delta t)}{(\Delta t)^2}. \\
 v(t_2) - v(t_1) &= \int_{t_1}^{t_2} a(t) dt.
 \end{aligned}$$

3.5.1 Motion with a constant acceleration

Free fall: $S(t) = \frac{1}{2}gt^2$ with $\begin{cases} S(t=0) = 0 \\ v(t=0) = 0 \end{cases}$

$$\left. \begin{aligned} v(t) &= \frac{ds(t)}{dt} = gt \\ a &= \frac{dv(t)}{dt} = g \end{aligned} \right\}$$

Or we consider a motion of constant acceleration

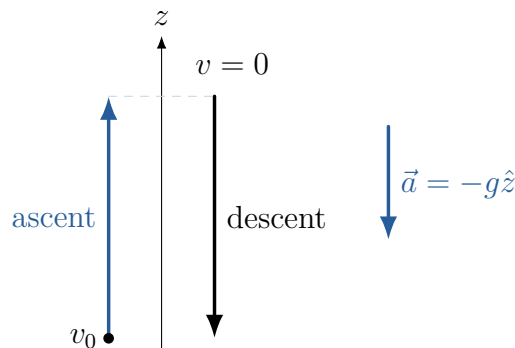
$$\frac{d^2 s}{dt^2} = g \quad \rightarrow \quad \frac{ds(t)}{dt} = gt + c_1 \rightarrow S(t) = \frac{1}{2}gt^2 + c_1 t + c_2$$

c_1 and c_2 are constants determined by the initial conditions.

If $S(t=0) = 0, v(t=0) = 0 \Rightarrow c_1 = c_2 = 0$.

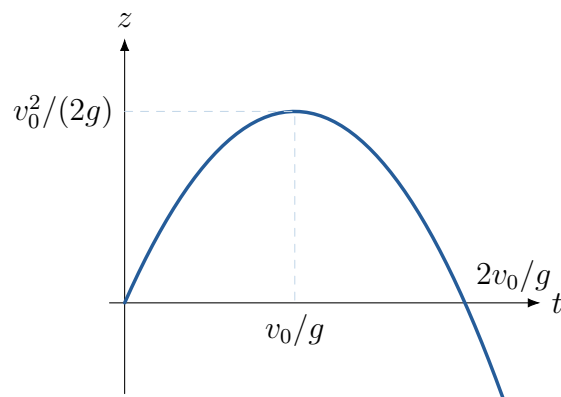
3.6 Displacement and velocity as vectors

1D case: unification of the motion of ascending and descending parts.



$$\begin{cases} \frac{d^2 z(t)}{dt^2} = -g, \\ \left. \frac{dz}{dt} \right|_{t=0} = v_0, \\ z(t=0) = 0. \end{cases}$$

$$\begin{aligned} \Rightarrow z(t) &= -\frac{1}{2}gt^2 + v_0 t \\ &= -\frac{g}{2} \left(t - \frac{v_0}{g} \right)^2 + \frac{v_0^2}{2g} \\ v(t) &= v_0 - gt \end{aligned}$$

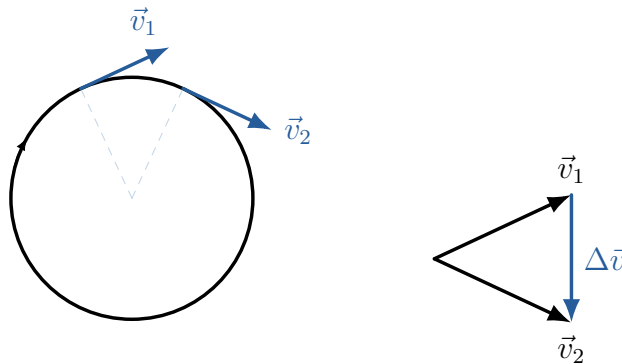
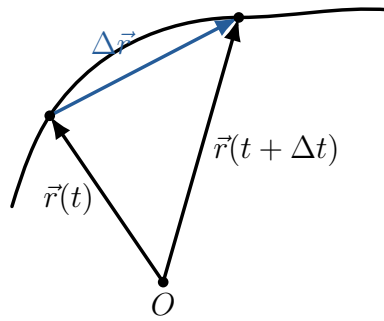


3.6.1 Motion in high-dimensions

We use the vector notation to represent displacement, $\vec{r}$, velocity $\vec{v}$, and acceleration $\vec{a}$. The corresponding concepts can be generalized from the straight-line motions

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \\ \vec{a} &= \frac{d\vec{v}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) + \vec{r}(t - \Delta t) - 2\vec{r}(t)}{(\Delta t)^2} \end{aligned}$$

* Circular motion with **uniform speed**, but it has the **centripetal acceleration**!

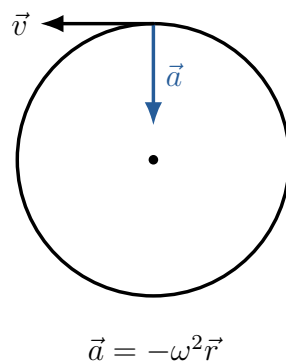


* We can write down the definition in terms of components:

$$\begin{aligned} v_x &= \frac{dx}{dt}, & a_x &= \frac{dv_x}{dt} = \frac{d^2x}{dt^2} \\ v_y &= \frac{dy}{dt}, & a_y &= \frac{dv_y}{dt} = \frac{d^2y}{dt^2} \\ v_z &= \frac{dz}{dt}, & a_z &= \frac{dv_z}{dt} = \frac{d^2z}{dt^2} \\ v &= \sqrt{v_x^2 + v_y^2 + v_z^2}, & a &= \sqrt{a_x^2 + a_y^2 + a_z^2}. \end{aligned}$$

3.6.2 Example: uniform speed circular motion

$$\begin{aligned} x &= r \cos \omega t, & y &= r \sin \omega t, \\ v_x &= -\omega r \sin \omega t, & v_y &= \omega r \cos \omega t, \\ a_x &= -\omega^2 r \cos \omega t, & a_y &= -\omega^2 r \sin \omega t. \end{aligned}$$



$$\Rightarrow \vec{a} = -\omega^2 \vec{r}$$

3.6.3 Example: projectile motion

$$v_x = v \cos \theta, \quad v_y = v \sin \theta$$

$$\frac{dx}{dt} = v_x, \quad \frac{dy}{dt} = v_y - gt$$

$$\Rightarrow \begin{aligned} x &= v_x t + C_1, \\ y &= v_y t - \frac{1}{2}gt^2 + C_2 \end{aligned}$$

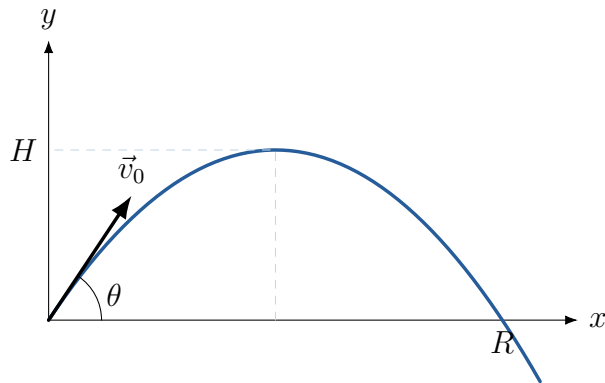
$$\text{with } x(0) = 0, \quad y(0) = 0$$

$$\Rightarrow C_1 = 0, \quad C_2 = 0$$

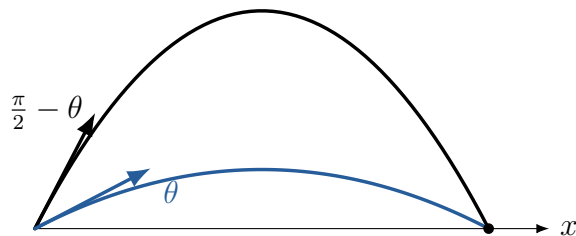
$$\Rightarrow \begin{aligned} x &= v \cos \theta t, \\ y &= v \sin \theta t - \frac{1}{2}gt^2 \end{aligned}$$

Equation of trajectory:

$$\begin{aligned} y &= x \tan \theta - \frac{g}{2v^2 \cos^2 \theta} x^2 \\ &= -\frac{g}{2v^2 \cos^2 \theta} \left(x - \frac{v^2 \sin 2\theta}{2g} \right)^2 + \frac{(v \sin \theta)^2}{2g} \end{aligned}$$



- Range of shooting: $\frac{\sin 2\theta}{g} v^2$
- Shooting altitude: $\frac{(v \sin \theta)^2}{2g}$



Same v_0 , same range: $R = v_0^2 \sin(2\theta)/g$.

3.7 Motion in polar coordinates

$$\hat{e}_r = \hat{x} \cos \theta + \hat{y} \sin \theta,$$

$$\hat{e}_\theta = -\hat{x} \sin \theta + \hat{y} \cos \theta.$$

$$\begin{aligned} \frac{d\hat{e}_r}{d\theta} &= -\hat{x} \sin \theta + \hat{y} \cos \theta, \\ &= \hat{e}_\theta, \end{aligned}$$

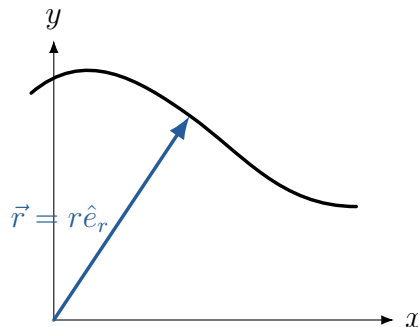
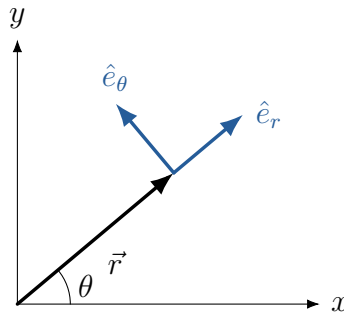
$$\begin{aligned} \frac{d\hat{e}_\theta}{d\theta} &= -\hat{x} \cos \theta - \hat{y} \sin \theta, \\ &= -(\hat{x} \cos \theta + \hat{y} \sin \theta) \\ &= -\hat{e}_r. \end{aligned}$$

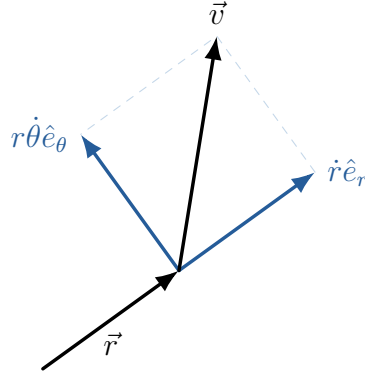
$$\begin{aligned} \frac{d\hat{e}_r}{dt} &= \frac{d\hat{e}_r}{d\theta} \frac{d\theta}{dt} = \hat{e}_\theta \frac{d\theta}{dt}, \\ \frac{d\hat{e}_\theta}{dt} &= \frac{d\hat{e}_\theta}{d\theta} \frac{d\theta}{dt} = -\hat{e}_r \frac{d\theta}{dt}. \end{aligned}$$

such that,

$$\vec{r}(t) = r(t) \hat{e}_r(t),$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r \hat{e}_r) = \frac{dr}{dt} \hat{e}_r + r \frac{d\theta}{dt} \hat{e}_\theta.$$





$$\begin{aligned}
 \vec{a}(r) &= \frac{d\vec{v}}{dt} = \frac{dr}{dt} \frac{d\hat{e}_r}{dt} + \hat{e}_r \frac{d^2r}{dt^2} \\
 &\quad + \frac{d\hat{e}_\theta}{dt} r \frac{d\theta}{dt} + \hat{e}_\theta \frac{dr}{dt} \frac{d\theta}{dt} + \hat{e}_\theta r \frac{d^2\theta}{dt^2} \\
 &= \left(2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right) \hat{e}_\theta + \left(\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right) \hat{e}_r \\
 &= \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \hat{e}_r + \frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) \hat{e}_\theta
 \end{aligned}$$

For circular motion with fixed r ,

$$\Rightarrow \vec{a} = -r \left(\frac{d\theta}{dt} \right)^2 \hat{e}_r + r \frac{d^2\theta}{dt^2} \hat{e}_\theta$$

First term is centripetal acceleration, and second term is the acceleration along tangential direction

3.8 Acceleration in the tangent direction and the normal direction

tangential (切向): 沿轨迹的切线; *normal* (法向): 垂直于切线; *curvature* (曲率): 描述曲线弯曲程度, 此处为 $1/R$.

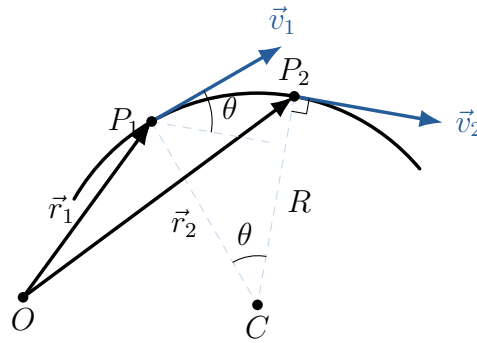
Consider a motion along a curve. Here $\hat{e}_t$ points along the velocity and $\hat{e}_n$ is the outward normal; $\Delta v_\perp$ denotes the inward component.

At $\vec{r}_1$, its velocity is $\vec{v}_1$,

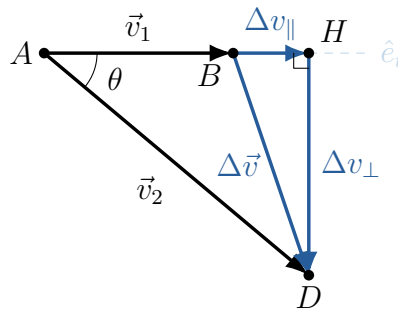
At $\vec{r}_2$, its velocity is $\vec{v}_2$.

The angle difference between $\vec{v}_1$ and $\vec{v}_2$ is θ .

$$\Rightarrow \Delta \vec{v} = \vec{v}_2 - \vec{v}_1 = \Delta v_\parallel \hat{e}_t - \Delta v_\perp \hat{e}_n.$$

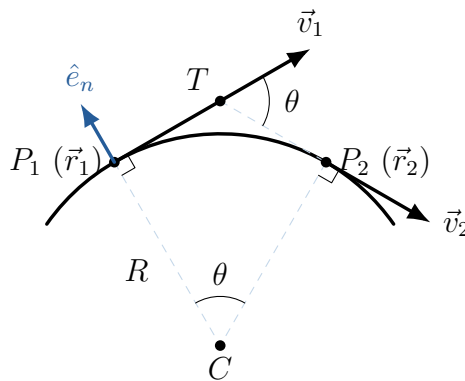


Here P_1 and P_2 are the two positions, O is the position-vector origin, and C is the local centre of curvature. Translating the velocities to a common origin gives the following velocity triangle; H is the perpendicular projection onto the initial tangent.



$$\Rightarrow \quad \vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{dv}{dt} \hat{e}_t - v \frac{d\theta}{dt} \hat{e}_n.$$

We approximate the short trajectory from $\vec{r}_1$ to $\vec{r}_2$ by a circular arc of radius R . The extended tangents intersect at T ; their turning angle equals the angle between the radii.



Here $1/R$ is the curvature.

Then,

$$\Rightarrow \vec{a} = \underbrace{\frac{dv}{dt} \hat{e}_t}_{\text{tangential acceleration} \quad \text{切向加速度}} + \underbrace{\left(-\frac{v^2}{R} \hat{e}_n \right)}_{\text{normal acceleration} \quad \text{法向加速度}}.$$

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