

Lecture 2: Vectors, scalar and cross products, rotations

Outline

1. Definition of a vector and the parallelogram law
2. Inner product
3. Cross product and directed area
4. Frame and coordinate transformations
5. Invariance of the scalar product

1 Vectors and physical quantities

Many physical quantities have both direction and magnitude and are represented as vectors. Displacement, velocity, acceleration, and force are examples. Pictorially, a vector can be represented by a directed line segment.

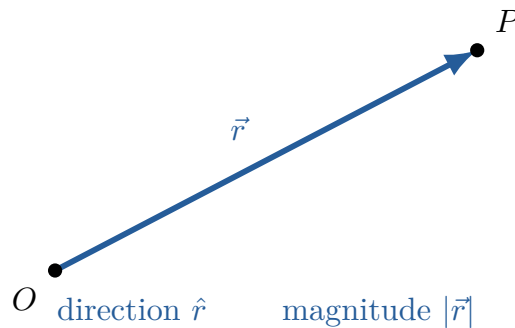


Figure 1: A vector from O to P . Its arrow shows the direction and its length represents the magnitude.

The direction of $\vec{r}$ is often denoted by $\hat{r}$. The magnitude of $\hat{r}$ is 1, so $\hat{r}$ is called a unit vector. The vector $-\vec{r}$ has the opposite direction but the same magnitude as $\vec{r}$; hence

$$\vec{r} + (-\vec{r}) = \vec{r} - \vec{r} = \vec{0}.$$

2 Vector addition and subtraction

The sum of two vectors is obtained by placing them head to tail, or equivalently by completing a parallelogram:

$$\vec{r} = \vec{r}_1 + \vec{r}_2.$$

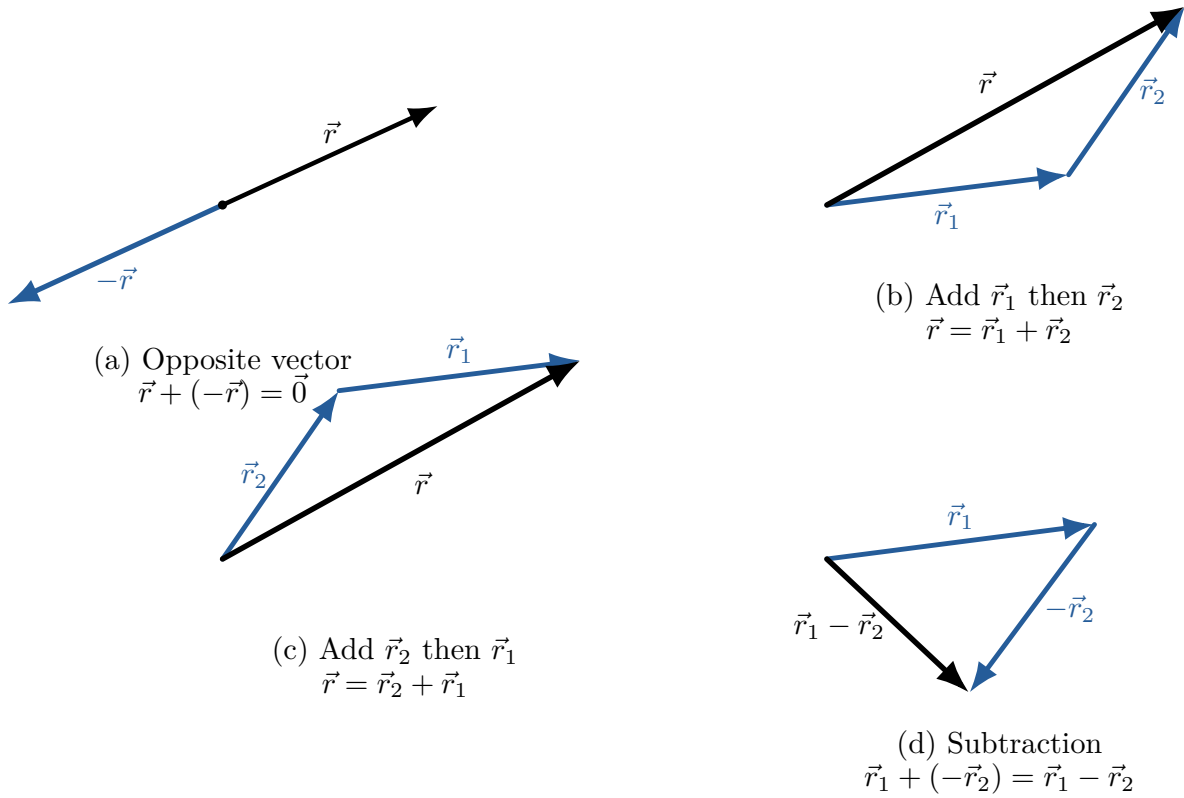


Figure 2: Opposite vectors, head-to-tail addition in both orders, and subtraction as addition of the opposite vector.

The order of addition does not change the result,

$$\vec{r}_2 + \vec{r}_1 = \vec{r}_1 + \vec{r}_2,$$

and subtraction is addition of the opposite vector,

$$\vec{r}_1 + (-\vec{r}_2) = \vec{r}_1 - \vec{r}_2.$$

3 Components and coordinate descriptions

Let $\hat{x}$, $\hat{y}$, and $\hat{z}$ be a set of mutually orthogonal unit vectors. They define a Cartesian coordinate system.

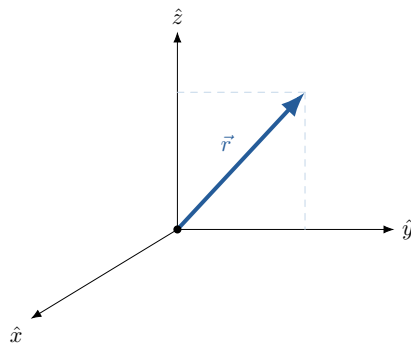


Figure 3: A vector resolved along three orthogonal basis directions.

A vector $\vec{r}$ is written as

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z},$$

where x , y , and z are its components. The magnitude of $\vec{r}$ is denoted by r or $|\vec{r}|$:

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}.$$

4 Inner product

Consider two vectors

$$\vec{r}_1 = x_1\hat{x} + y_1\hat{y} + z_1\hat{z}, \quad \vec{r}_2 = x_2\hat{x} + y_2\hat{y} + z_2\hat{z}.$$

Their inner product, also called the scalar product, is defined as

$$\vec{r}_1 \cdot \vec{r}_2 = x_1x_2 + y_1y_2 + z_1z_2.$$

In particular,

$$r^2 = \vec{r} \cdot \vec{r}.$$

The orthonormal basis vectors obey

$$\begin{aligned} \hat{x} \cdot \hat{x} &= \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1, \\ \hat{x} \cdot \hat{y} &= \hat{y} \cdot \hat{x} = 0, \quad \hat{x} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0, \quad \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{y} = 0. \end{aligned}$$

4.1 Geometrical meaning

Let θ be the angle between $\vec{r}_1$ and $\vec{r}_2$. The law of cosines gives

$$\begin{aligned} |\vec{r}_2 - \vec{r}_1|^2 &= (\vec{r}_2 - \vec{r}_1) \cdot (\vec{r}_2 - \vec{r}_1) \\ &= \vec{r}_2 \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_1 - 2\vec{r}_1 \cdot \vec{r}_2 \\ &= r_2^2 + r_1^2 - 2r_1r_2 \cos \theta. \end{aligned}$$

Therefore,

$$\boxed{\vec{r}_1 \cdot \vec{r}_2 = r_1r_2 \cos \theta}.$$

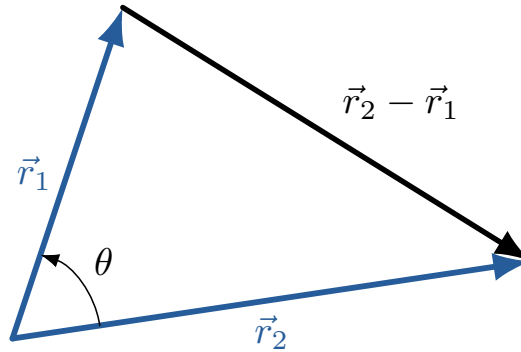


Figure 4: The triangle used to identify the geometrical meaning of the inner product.

5 Applications of the inner product

5.1 Equation of a plane

Let $\vec{ON}$ be normal to a plane, with its endpoint N on the plane. If $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ is the position vector of an arbitrary point on the plane, then

$$\vec{r} \cdot \vec{ON} = |\vec{ON}|^2.$$

Writing

$$\vec{ON} = N_x\hat{x} + N_y\hat{y} + N_z\hat{z},$$

the plane equation becomes

$$xN_x + yN_y + zN_z = |\vec{ON}|^2,$$

or

$$\frac{xN_x}{|\vec{ON}|^2} + \frac{yN_y}{|\vec{ON}|^2} + \frac{zN_z}{|\vec{ON}|^2} = 1.$$

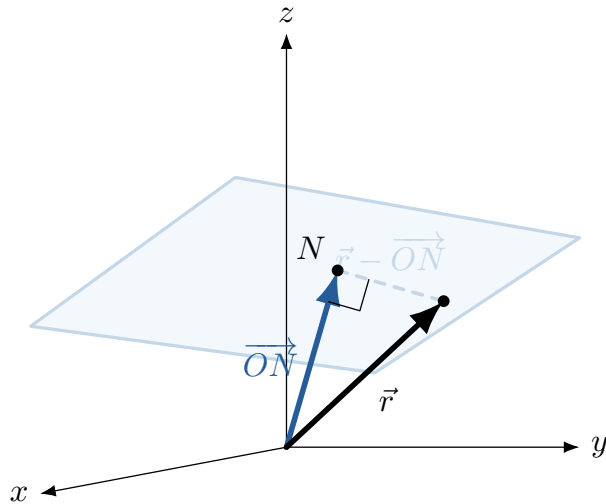


Figure 5: A plane specified by the normal vector $\vec{ON}$. The dashed in-plane segment joins N to the endpoint of $\vec{r}$.

5.2 Work and power

For a constant force $\vec{F}$ and displacement $\vec{r}$,

$$W = Fr \cos \theta = \vec{F} \cdot \vec{r}.$$

The instantaneous power is

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}.$$

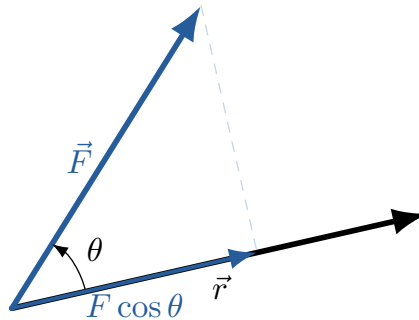


Figure 6: Only the component of force parallel to the displacement contributes to the work.

5.3 Volume swept by a directed area

Represent an area of magnitude S and unit normal $\hat{n}$ by the directed area

$$\vec{S} = S\hat{n}.$$

If the area is swept through a displacement $\vec{l}$, the volume is

$$V = \vec{S} \cdot \vec{l} = S\hat{n} \cdot \vec{l}.$$

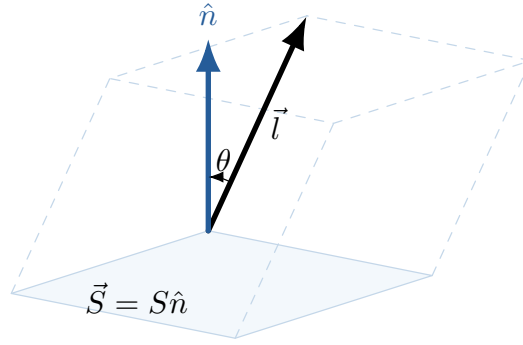


Figure 7: A directed area $\vec{S}$ swept through a displacement $\vec{l}$; $\hat{n}$ and $\vec{l}$ share an origin and enclose the angle θ .

6 Cross product

The cross product of the Cartesian basis vectors is defined by

$$\begin{aligned}\hat{x} \times \hat{x} &= \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0, \\ \hat{x} \times \hat{y} &= \hat{z}, & \hat{y} \times \hat{z} &= \hat{x}, & \hat{z} \times \hat{x} &= \hat{y}, \\ \hat{y} \times \hat{x} &= -\hat{z}, & \hat{z} \times \hat{y} &= -\hat{x}, & \hat{x} \times \hat{z} &= -\hat{y}.\end{aligned}$$

The direction follows the right-hand rule.

For two vectors in the xy plane,

$$\vec{A} = A_x \hat{x} + A_y \hat{y}, \quad \vec{B} = B_x \hat{x} + B_y \hat{y},$$

as shown below.

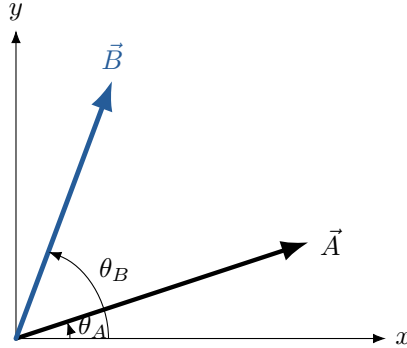


Figure 8: Two vectors in the xy plane, measured from the positive x direction.

Algebraically, we obtain

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x B_y - A_y B_x) \hat{z}, \\ A_x &= |\vec{A}| \cos \theta_A, & A_y &= |\vec{A}| \sin \theta_A, \\ B_x &= |\vec{B}| \cos \theta_B, & B_y &= |\vec{B}| \sin \theta_B,\end{aligned}$$

and therefore

$$\begin{aligned}\vec{A} \times \vec{B} &= |\vec{A}| |\vec{B}| (\cos \theta_A \sin \theta_B - \sin \theta_A \cos \theta_B) \hat{z} \\ &= |\vec{A}| |\vec{B}| \sin(\theta_B - \theta_A) \hat{z}.\end{aligned}$$

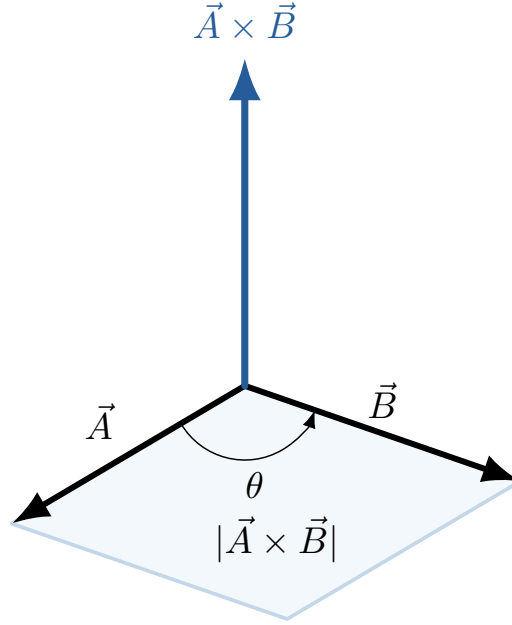


Figure 9: The cross product is perpendicular to the plane of $\vec{A}$ and $\vec{B}$; its magnitude is the area of their parallelogram.

For general three-dimensional vectors,

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}, \quad \vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z},$$

the cross product is

$$\begin{aligned} \vec{C} = \vec{A} \times \vec{B} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x). \end{aligned}$$

The components follow a cyclic pattern. Direct substitution shows

$$\vec{C} \cdot \vec{A} = 0, \quad \vec{C} \cdot \vec{B} = 0,$$

so $\vec{C}$ is perpendicular to the plane spanned by $\vec{A}$ and $\vec{B}$.

The magnitude satisfies

$$\begin{aligned} |\vec{C}|^2 &= (A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2 + (A_x B_y - A_y B_x)^2 \\ &= |\vec{A}|^2 |\vec{B}|^2 - |\vec{A} \cdot \vec{B}|^2 \\ &= |\vec{A}|^2 |\vec{B}|^2 \sin^2 \theta. \end{aligned}$$

Thus

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta, \quad \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}.$$

7 Scalar triple product

For a parallelepiped with edge vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, the directed base area is $\vec{A} \times \vec{B}$. Its volume is

$$V = (\vec{A} \times \vec{B}) \cdot \vec{C}.$$

The same volume can be evaluated using any face as the base. The scalar triple product is invariant under cyclic permutations:

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B}.$$

Exchanging two vectors changes the sign:

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = -(\vec{B} \times \vec{A}) \cdot \vec{C} = -(\vec{C} \times \vec{B}) \cdot \vec{A} = -(\vec{A} \times \vec{C}) \cdot \vec{B}.$$

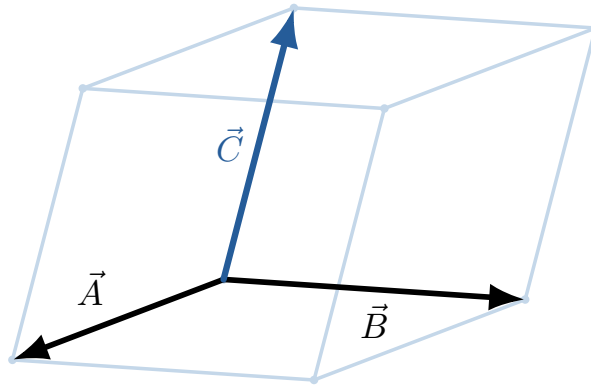


Figure 10: The scalar triple product gives the oriented volume of the complete eight-vertex parallelepiped.

8 Physical examples of cross products

8.1 Angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}.$$

8.2 Torque

$$\vec{N} = \vec{r} \times \vec{F}.$$

8.3 Lorentz force

$$\vec{F} = \frac{q}{c} \vec{v} \times \vec{B} \quad (\text{Gaussian units}), \quad \vec{F} = q \vec{v} \times \vec{B} \quad (\text{SI units}).$$

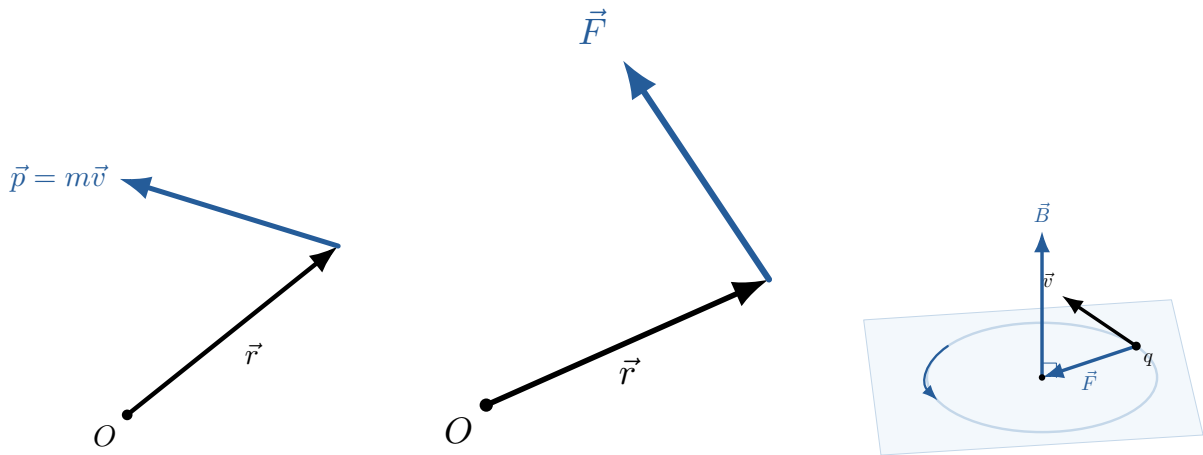


Figure 11: Three physical quantities whose directions are fixed by a cross product. From left to right: angular momentum with $\vec{p} = m\vec{v}$, torque, and magnetic force providing the centripetal force for circular motion.

9 Coordinate frames and passive rotations

Consider two frames with basis vectors $(\hat{x}', \hat{y}', \hat{z}')$ and $(\hat{x}, \hat{y}, \hat{z})$, with $\hat{z}' = \hat{z}$. Let the primed basis be rotated counterclockwise through an angle θ relative to the unprimed basis:

$$\begin{aligned} \hat{x}' &= \hat{x} \cos \theta + \hat{y} \sin \theta, \\ \hat{y}' &= -\hat{x} \sin \theta + \hat{y} \cos \theta. \end{aligned}$$

Define

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Then

$$(\hat{x}', \hat{y}') = (\hat{x}, \hat{y})R(\theta).$$

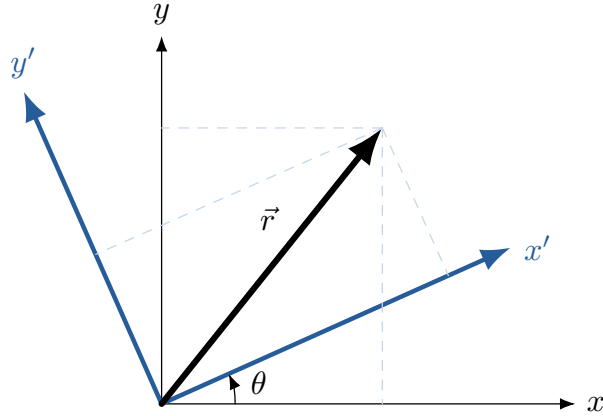


Figure 12: Passive rotation of the coordinate basis. The geometrical vector $\vec{r}$ is held fixed while its projections change.

An $m \times n$ matrix A has entries A_{ij} , where i selects a row and j selects a column.

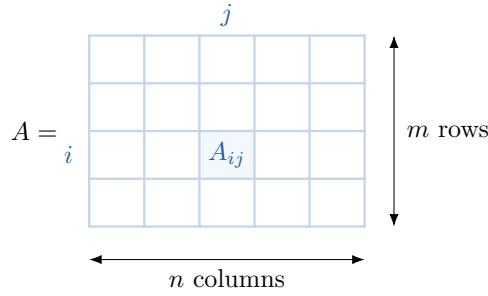


Figure 13: Row and column indexing of a matrix element A_{ij} .

For matrix multiplication $C = AB$, the element C_{ik} is the inner product of row i of A with column k of B . Reversing the order generally changes the product.

$$C_{ik} = \underbrace{\begin{pmatrix} A_{i1} & A_{i2} & \cdots & A_{in} \end{pmatrix}}_{\text{row } i \text{ of } A} \underbrace{\begin{pmatrix} B_{1k} \\ B_{2k} \\ \vdots \\ B_{nk} \end{pmatrix}}_{\text{column } k \text{ of } B} = \sum_j A_{ij} B_{jk}.$$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

$$BA = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}.$$

$AB \neq BA \quad \text{in general.}$

Figure 14: Row-by-column matrix multiplication and the noncommutativity of two general 2×2 matrices.

The transpose is defined by

$$(A^T)_{ij} = A_{ji}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

The same vector may be expanded in either basis:

$$\vec{r} = x\hat{x} + y\hat{y} = x'\hat{x}' + y'\hat{y}'.$$

Therefore,

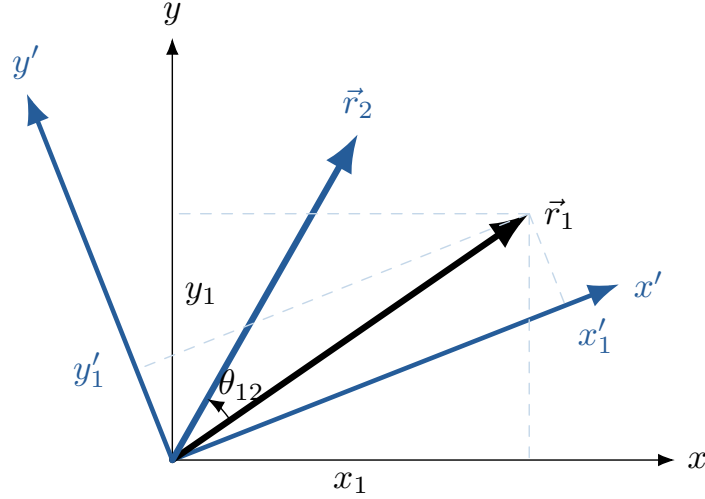
$$\begin{pmatrix} x \\ y \end{pmatrix} = R(\theta) \begin{pmatrix} x' \\ y' \end{pmatrix},$$

and

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\theta)^\top \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

10 Invariance of the inner product

The same geometrical vectors may be described using either the unprimed or primed coordinate basis. Their components change, while their relative angle and inner product remain fixed.



$$\begin{aligned} x_1 x_2 + y_1 y_2 &= x'_1 x'_2 + y'_1 y'_2 \\ \mathbf{r}_1^\top R R^\top \mathbf{r}_2 &= \mathbf{r}_1^\top I \mathbf{r}_2 \end{aligned}$$

Figure 15: Two fixed vectors described in rotated coordinate frames. The component expressions reduce to the same inner product.

In the unprimed basis,

$$\vec{r}_1 \cdot \vec{r}_2 = x_1 x_2 + y_1 y_2 = \begin{pmatrix} x_1 & y_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

In the primed basis,

$$\vec{r}_1 \cdot \vec{r}_2 = x'_1 x'_2 + y'_1 y'_2 = \begin{pmatrix} x'_1 & y'_1 \end{pmatrix} \begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix}.$$

Since the coordinate columns transform as $\mathbf{r}' = R^\top \mathbf{r}$,

$$\begin{aligned} (\mathbf{r}'_1)^\top \mathbf{r}'_2 &= (R^\top \mathbf{r}_1)^\top (R^\top \mathbf{r}_2) \\ &= \mathbf{r}_1^\top R R^\top \mathbf{r}_2 \\ &= \mathbf{r}_1^\top \mathbf{r}_2, \end{aligned}$$

because

$$R R^\top = I.$$

This agrees with the geometrical expression

$$\vec{r}_1 \cdot \vec{r}_2 = |\vec{r}_1| |\vec{r}_2| \cos \theta_{12},$$

which is independent of the choice of coordinate frame. This invariance is why the inner product is called a scalar product.

11 Active rotations

In an active rotation, the coordinate frame is held fixed while the vector is rotated. If

$$x = r \cos \alpha, \quad y = r \sin \alpha,$$

then a counterclockwise rotation through θ gives

$$\begin{aligned} x' &= r \cos(\theta + \alpha) = \cos \theta x - \sin \theta y, \\ y' &= r \sin(\theta + \alpha) = \sin \theta x + \cos \theta y. \end{aligned}$$

Thus

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\theta) \begin{pmatrix} x \\ y \end{pmatrix}.$$

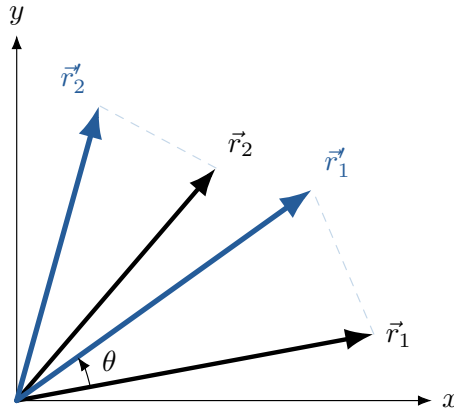


Figure 16: Active rotation of a pair of vectors through the same angle θ . Their relative angle is unchanged.

If a pair of vectors is rotated by the same matrix R , then

$$\begin{aligned} \vec{r}_1' \cdot \vec{r}_2' &= (R\mathbf{r}_1)^\top (R\mathbf{r}_2) \\ &= \mathbf{r}_1^\top R^\top R \mathbf{r}_2 \\ &= \vec{r}_1 \cdot \vec{r}_2. \end{aligned}$$

The identity

$$R^\top R = I$$

defines an orthogonal matrix. It expresses the preservation of lengths and angles under a rotation.