

Homework 6

Due: November 3, 2026

Code number: _____

Score: _____

Problem 1: The recoil-energy distribution in elastic collisions

An incident ball of mass $m_1 > 0$ collides elastically with a target ball of mass $m_2 > 0$, initially at rest in the laboratory frame. Every incident ball has the same initial kinetic energy

$$E_0 = \frac{1}{2}m_1v_0^2 > 0.$$

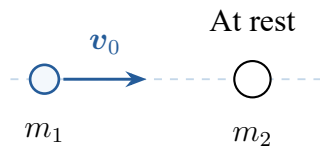
Consider nonrelativistic motion in three dimensions. Neglect external impulses, rotation, and internal excitation. Let E_R be the target's kinetic energy after the collision, measured in the laboratory frame.

Let θ^* be the angle between the incident ball's initial and final momenta in the center-of-mass (CM) frame. Assume its outgoing direction is isotropic in this frame: equal solid angles have equal probability,

$$dP = \frac{d\Omega^*}{4\pi}, \quad d\Omega^* = \sin\theta^* d\theta^* d\phi^*,$$

with $0 \leq \theta^* \leq \pi$ and $0 \leq \phi^* < 2\pi$.

(A) Laboratory frame: before



(B) CM momentum vectors

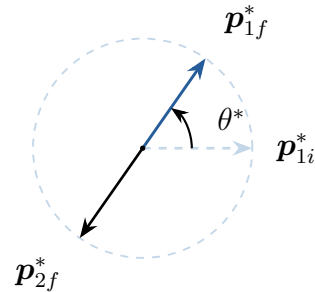


Fig. 1: The initial laboratory configuration and a CM momentum diagram. In (B), the solid arrows are the final momenta and the dashed arrow is the incident ball's initial momentum. One scattering plane is shown; isotropy refers to directions over the full solid angle.

1. Use conservation of momentum and kinetic energy to find E_R as a function of θ^* . Determine the maximum recoil energy $E_{R,\max}$ in terms of m_1 , m_2 , and E_0 .
2. Starting from the solid-angle probability above, find the normalized probability density $f(E_R)$, defined by $dP = f(E_R) dE_R$. State its allowed energy range, sketch the distribution, and find the mean recoil energy $\langle E_R \rangle$.

Problem 2: Kepler orbits

A particle of mass m moves around a fixed attracting center under the force

$$\mathbf{F} = -\frac{\gamma}{r^2} \mathbf{e}_r, \quad \gamma > 0.$$

Take the potential energy to be zero at infinity. Let $E < 0$ and $\ell > 0$ be the particle's total mechanical energy and angular-momentum magnitude. Its orbit is a nondegenerate ellipse with semimajor axis a , semiminor axis b , eccentricity e , and semi-latus rectum c . Here c is half the chord through the attracting focus perpendicular to the major axis, as shown below.

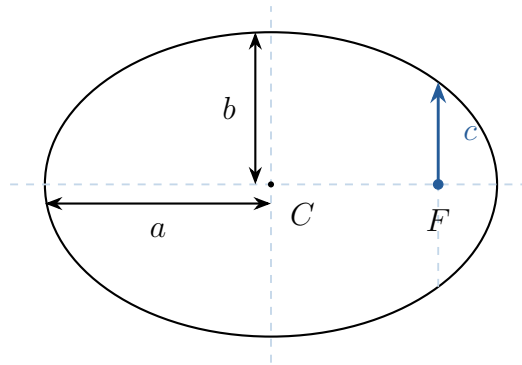


Fig. 2: An elliptical orbit with center C and attracting focus F . The lengths a , b , and c are shown.

You may use the polar orbit equation

$$r = \frac{c}{1 + e \cos \phi},$$

where the origin is at F and $\phi = 0$ points toward periapsis.

1. Express a , b , c , and e in terms of E , ℓ , m , and γ . Identify the condition on E and ℓ for a bound orbit to exist.
2. Show that a depends only on E , while c depends only on ℓ for fixed m and γ .
3. For a fixed negative energy E , find the largest possible ℓ . For a fixed $\ell > 0$, find the lowest possible E . Identify the orbit in each case and explain why the extremum occurs there.
4. Use the area of the ellipse and the areal velocity to derive Kepler's third law, including the proportionality constant in $T^2 \propto a^3$.

Problem 3: The three cosmic velocities

Model Earth and the Sun as spherical bodies with masses M_E and M_S . Earth has radius R_E and follows a circular orbit of radius a_E around the Sun. A spacecraft has negligible mass. Ignore Earth's rotation, atmospheric drag, other planets, and propulsion after launch.

Use the following approximation: near Earth, treat the motion as an Earth–spacecraft two-body problem; after the spacecraft escapes Earth, treat its subsequent motion as a Sun–spacecraft two-body problem. During Earth escape, neglect the change in Earth's heliocentric position and orbital velocity. The asymptotic Earth-relative velocity is then added to Earth's orbital velocity to obtain the spacecraft's heliocentric velocity.

Use

$$\begin{aligned} G &= 6.6743 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}, \\ M_E &= 5.972 \times 10^{24} \text{ kg}, & R_E &= 6.371 \times 10^6 \text{ m}, \\ M_S &= 1.989 \times 10^{30} \text{ kg}, & a_E &= 1.496 \times 10^{11} \text{ m}. \end{aligned}$$

Give all numerical speeds in km s^{-1} .

1. The **first cosmic velocity** v_1 is the speed of a circular orbit just above Earth's surface. Derive its expression and calculate its value.
2. The **second cosmic velocity** v_2 is the minimum launch speed from Earth's surface needed to escape Earth's gravity with zero speed at infinity in the Earth-centered model. Derive its expression and calculate its value. Both v_1 and v_2 are measured relative to Earth.
3. The **third cosmic velocity** v_3 is the minimum Earth-relative launch speed from Earth's surface needed to escape the Solar System in this approximation.
 - (a) Find Earth's heliocentric orbital speed V_E and the solar escape speed at $r = a_E$.
 - (b) Let $\mathbf{u}_\infty$ be the spacecraft's asymptotic velocity relative to Earth after Earth escape. Use

$$\mathbf{V} = \mathbf{V}_E + \mathbf{u}_\infty$$

to find the direction and minimum magnitude of $\mathbf{u}_\infty$ needed for solar escape.

- (c) Use conservation of energy in the Earth-centered escape problem to relate v_3 to u_∞ and v_2 . Calculate v_3 .

Problem 4: Changing an asteroid's orbit with DART

NASA's DART spacecraft struck Dimorphos, the small moon of the asteroid Didymos, to test asteroid deflection. Consider the following simplified model. Treat Didymos as a fixed point mass M and Dimorphos as a point mass m in an initially circular orbit with period T_0 . Ignore the Sun's gravity and the spin and shape of both asteroids. Use

$$M = 5.28 \times 10^{11} \text{ kg}, \quad m = 4.8 \times 10^9 \text{ kg},$$

$$T_0 = 11 \text{ h } 55 \text{ min}, \quad m_s = 600 \text{ kg}, \quad v_{\text{imp}} = 6.26 \text{ km s}^{-1}.$$

Here m_s is the spacecraft mass and v_{imp} is its speed *relative to Dimorphos immediately before impact*. For parts 1–3, assume the spacecraft sticks to Dimorphos and no material is ejected. The collision is instantaneous, so the orbital position is unchanged during impact. You may neglect m_s compared with m in the subsequent orbital dynamics. Use the value of G given in Problem 3.

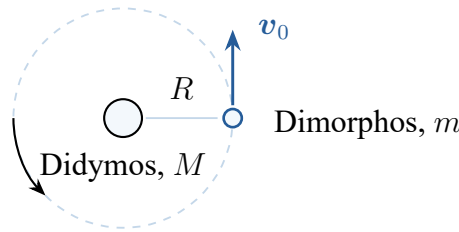


Fig. 3: The circular orbit before impact. The incoming spacecraft is omitted so that its direction can be chosen in part 2.

1. Find the initial orbital radius R and orbital speed v_0 of Dimorphos.
2. Choose a tangential impact opposite to Dimorphos's orbital motion. Use momentum conservation during the collision to find the speed v' just after impact. Explain why this direction reduces the orbital period more than a tangential impact along the orbital motion at the same relative impact speed.
3. The new orbit is elliptical. Find its semimajor axis a' from the post-impact orbital energy, then calculate its period T' and the period reduction $\Delta T = T_0 - T'$. You may use your results from Problem 2 with $\gamma = GMm$.
4. NASA's minimum success criterion was a period reduction of 73 s; observations found a reduction of about 33 min.¹ Compare your prediction with these values. Material was ejected during the real impact. Explain qualitatively how ejecta recoil could produce a larger reduction than the sticking model predicts, and state the required direction of the ejecta's net momentum relative to Dimorphos's orbital motion.

¹NASA, *Didymos & Dimorphos*.