

Homework 5

Due: October 27, 2026

Code number: _____

Score: _____

Problem 1: Reversible kinetic-energy machines

Consider a particle released from rest and lowered through a fixed vertical height h by a kinetic-energy machine, as discussed in Lecture 8. Compare machines acting on the same particle between the same upper and lower levels. Treat the particle as a point mass, with no rotational energy. Each machine can operate repeatedly, returning to its initial mechanical state after each passage. It receives no external power and has no net change in stored energy; an irreversible machine may release heat to its surroundings.

A machine is *reversible* if reversing the particle's velocity at the lower level makes it retrace the process and return to rest at the upper level, restoring the machine to its original state.

Use the impossibility of a perpetual-motion machine to prove the following statements. Describe how the machines could be combined to obtain a contradiction if a statement were false.

1. No machine can give the particle a greater final speed than a reversible machine with the same vertical drop.
2. Any two reversible machines with the same vertical drop give the particle the same final speed, regardless of their design.

Base your argument on reversibility and the impossibility of perpetual motion; do not assume the mechanical-energy conservation law that this argument is intended to motivate.

Problem 2: The energy of an interacting many-particle system

A system consists of N point particles with masses m_i , positions $\mathbf{r}_i$, and velocities $\mathbf{v}_i$. It is in a uniform gravitational field $-g\hat{\mathbf{z}}$, with z measured vertically upward.

The interaction between particles i and j is a time-independent central force depending only on their separation $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$:

$$\mathbf{F}_{ij} = f_{ij}(r_{ij}) \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}}, \quad f_{ij} = f_{ji},$$

where $\mathbf{F}_{ij}$ is the force on particle i due to particle j . Thus $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$. There are no other forces.

1. Introduce a pair potential energy $U_{ij}(r_{ij})$ and state its relation to $f_{ij}(r_{ij})$. Write the conserved total energy of the system, including kinetic energy, gravitational potential energy, and the interaction potential energies.

Explain why each interacting pair must be counted once. Give equivalent expressions for the interaction energy using $\sum_{i < j}$ and $\frac{1}{2} \sum_{i \neq j}$.

2. Starting from Newton's second law for each particle, show briefly that the time derivative of your total energy is zero. Show explicitly how the work done by the two forces in each interacting pair combines with the change in U_{ij} .

Problem 3: A cube balanced on a cylinder

A uniform cube of mass m and edge length $2b$ rests on top of a fixed horizontal cylinder of radius r , as shown in Fig. 1. Initially, the cube's center lies vertically above the cylinder's axis.

Consider small rocking motions in the plane of the figure, perpendicular to the cylinder's axis. Static friction is sufficient to prevent slipping, and the cube remains in contact with the cylinder along its bottom face. Take the gravitational acceleration to be g .

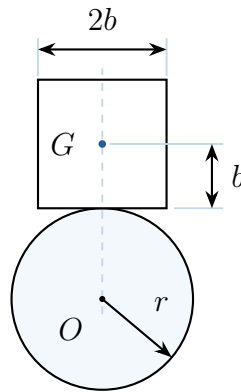


Fig. 1: The initial equilibrium. O is the cylinder's center in cross section, and G is the cube's center of mass.

Before calculating, use your intuition to predict whether increasing the cylinder's radius makes the equilibrium more or less stable.

1. Let the contact point move through a small angle θ from the top of the cylinder. Express the height $H(\theta)$ of the cube's center of mass above O .

Hint: Without slipping, the distance rolled along the cube's bottom face is $r\theta$.

2. Expand $H(\theta) - H(0)$ to second order in θ . Use the change in gravitational potential energy to find the condition for a positive quadratic term and hence a stable equilibrium. Compare with your initial prediction. The borderline case in which the quadratic term vanishes need not be analyzed.

Problem 4: The tidal potential and equilibrium ocean height

In Homework 4, the lunar tidal force was found to be, to first order in $|\mathbf{r}|/D$,

$$\mathbf{F}_{\text{tide}}(\mathbf{r}) = \frac{GMm}{D^3} [3\mathbf{n}(\mathbf{n} \cdot \mathbf{r}) - \mathbf{r}].$$

Here M is the Moon's mass, D is the Earth–Moon distance, $\mathbf{n}$ is a unit vector from Earth's center toward the Moon, and $\mathbf{r}$ is the position of a test mass m relative to Earth's center. Use the same translating, nonrotating Earth-centered frame as in Homework 4.

1. Find a tidal potential energy $U_{\text{tide}}(\mathbf{r})$ satisfying

$$\mathbf{F}_{\text{tide}} = -\nabla U_{\text{tide}}, \quad U_{\text{tide}}(\mathbf{0}) = 0.$$

Verify your result by taking its gradient.

2. Model Earth as a sphere of radius R , covered by an ocean in instantaneous equilibrium. Hold the Moon's direction fixed while finding the equilibrium surface. Neglect Earth's rotation, deformation of the solid Earth, and the ocean's self-gravity.

Let h be the small sea-surface displacement from the reference sphere, with $|h| \ll R$, and take the terrestrial gravitational acceleration g to be constant near the surface. The change in Earth's gravitational potential energy is then mgh . The free surface is an equipotential of the total potential energy.

To leading order in the tidal perturbation, evaluate U_{tide} at $\mathbf{r} = R\hat{\mathbf{r}}$ while retaining mgh for the terrestrial contribution. Find the height difference

$$\Delta h = h_{\parallel} - h_{\perp},$$

where $h_{\parallel}$ is the surface height along the Earth–Moon line and $h_{\perp}$ is the height in a perpendicular direction. Express your answer in terms of G , M , R , D , and g , and identify which direction has the higher surface.