HW10: Due on Dec 2nd in class. CODE NUMBER: ————— SCORE: —————

Problem #1: Virial Theorem

In class, we learned the Virial theorem, which can be written as:

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle$$

(a) Consider an ideal gas in 3D, from the equipartition theorem the average kinetic energy per atom in an ideal gas is 3kT/2, which is the left-hand side. Consider on the right-hand side the total force exerted by the ideal gas on the walls of the pressured vessel, where you can safely assume no interactions between atoms. Derive the ideal gas law from the above equation.

(b) The Virial theorem is a fundamental tool in astrophysics, especially for understanding the dynamics of systems in equilibrium, such as a cluster of galaxies. It provides a way to estimate the total mass of a cluster based on observable quantities. Consider the force of gravity, and derive the total mass of cluster M, as a function of the mean squared velocity $\langle \mathbf{v}^2 \rangle$ and cluster radius R.

Problem #2: Equal partition theorem & Maxwell distribution (double points)

Feynman Lecture Vol (I) Lecture 39 presents a beautiful explanation of the equal partition of energy among different degrees of freedom. The deep reason arises from the complete randomness, which Feynman would like to express but did not express it clearly. Here we examine it from a more general perspective.

Suppose we do not know the Maxwell or Boltzmann distributions as a priori. We will try to understand why it has to be the case.

There are billions of degrees of freedom in a gas. Let us pick up two of them, and express them in terms of two independent variables a_1, a_2. They are re-scaled, such that they contribute to the energy as

$$E = \frac{1}{2}(a_1^2 + a_2^2). \qquad (*)$$

For example, it could be $a = p/\sqrt{m}$ for the kinetic energy with p the momentum, or, $\sqrt{K}x$ for the elastic potential energy for intra-molecular vibration with K the spring constant. In the thermal equilibrium, we assume that only $\langle a_i^2 \rangle$ (i = 1, 2) can be nonzero, but the inter-variable correlation should vanish, i.e., $\langle a_1 a_2 \rangle = 0$, where $\langle \rangle$ means thermal average.

(a) Now let us reorganize these variables as

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

such that $E = \frac{1}{2}(b_1^2 + b_2^2)$. Prove that if we assume the determinant of the transformation matrix to be positive, the matrix element can be expressed in terms of an angle θ as

$$A = D = \cos\theta, \quad -B = C = \sin\theta.$$

In other words, it could be viewed as a rotation.

A collision between two atoms with non-equal masses m_1 and m_2 can also be decomposed into the center of mass motion and the relative motion. Figure out the transformation from $\mathbf{p_1}$ and $\mathbf{p_2}$ to the center mass momentum $\mathbf{P}$, the momentum $\mathbf{p'}$ of the relative motion, and the transformation angle.

(b) If the system is completely random, then we should have $\langle b_1 b_2 \rangle = 0$ for an arbitrary θ. Otherwise, there still exist certain correlations on a suitably chosen basis. Prove that this requires that $\langle a_i^2 \rangle = \langle a_i^2 \rangle$, which is equivalent to the equal partition theorem.

(c) The probability distribution function $\rho(a_1, a_2)da_1da_2$ gives rise to the relative probability for the system variables lying from a_1 to a_1+da_1 and a_2 to a_2+da_2. Since a_1 and a_2 are independent and are symmetric in Eq. (*), we arrive at

$$\rho(a_1, a_2) = f(a_1^2)f(a_2^2).$$

When expressed in variables b_1 and b_2, we should have

$$f(a_1^2)f(a_2^2) = f(b_1^2)f(b_2^2).$$

Prove that only the exponential functions can satisfy the above relation, i.e.,

$$f(a^2) = e^{-\frac{\beta}{2}a^2},$$

where β is a common parameter for all degrees of freedom.

(d) We define the temperature according to $\frac{1}{2}\langle a^2\rangle = k_B T$. Based on this definition and properties of the Gaussian integral, prove that $\beta = (k_B T)^{-1}$.

Problem #3: Atoms escaping a furnace

A furnace at temperature T contains n atoms of mass m per unit volume, some of which escape through a small hole of area A. (The hole is small enough so that equilibrium inside the oven is not disturbed).

(a) What is the number dn of atoms leaving through the area A per second, having speeds between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$?

(b) What is the root mean square velocity $\mathbf{v}^{(e)}_{rms}$ of the escaping atoms? (If different from the root mean square velocity inside the furnace, $\mathbf{v}^{(i)}_{rms}$, explain why.)

Problem #4: Blackbody radiation

The distribution law for black-body radiation is:

$$I(\omega) = \frac{\hbar\omega^3 d\omega}{\pi^2 c^2 \left(e^{\hbar\omega/kT} - 1\right)}$$

By changing the variable from ω to $z = \hbar\omega/kT$, show that:

(a) The total radiation intensity, integrated over all frequencies, is proportional to the fourth power of the absolute temperature.

(b) The frequency ω_m at which $I(\omega)$ has its maximum value is proportional to the absolute temperature.