

General Physics I: Homework 2

Tutorial Handout

Contents

1	Problem 1: Motion under a position-dependent force	2
2	Problem 2: Motion in crossed electric and magnetic fields	2
3	Problem 3: The region accessible to a projectile	5
4	Problem 4: Slowing down after a parachute opens	6
5	Problem 5: The longest fall	7

1 Problem 1: Motion under a position-dependent force

Strategy

Use the chain rule to replace the time derivative by a position derivative. This turns Newton's law into an equation for $v^2(x)$, which can be integrated directly.

Derivation

On a segment where v is differentiable as a function of x ,

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx} = \frac{1}{2} \frac{d(v^2)}{dx}.$$

Thus $m dv/dt = F(x)$ becomes

$$\frac{m}{2} \frac{d(v^2)}{dx} = F(x).$$

Integrating from (x_0, v_0) to x gives

$$m \int_{v_0}^{v(x)} v' dv' = \frac{m}{2} [v'^2]_{v_0}^{v(x)} = \frac{m}{2} (v^2(x) - v_0^2) = \int_{x_0}^x F(x') dx'.$$

hence

$$v^2(x) = v_0^2 + \frac{2}{m} \int_{x_0}^x F(x') dx'.$$

At a turning point $v = 0$, but v^2 is continuous, so the relation extends by continuity. It determines v^2 , not the sign of v ; the sign follows from the direction of travel.

For constant $F(x) = F_0$, $a = F_0/m$ and

$$v^2 = v_0^2 + \frac{2F_0}{m}(x - x_0) = v_0^2 + 2a(x - x_0),$$

the usual constant-acceleration relation. Since $W = \int_{x_0}^x F(x') dx'$, this is also the work-energy theorem:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = W.$$

2 Problem 2: Motion in crossed electric and magnetic fields

A positive charge q moves in $\mathbf{E} = E\hat{\mathbf{y}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$, where $E, B > 0$. It starts at the origin with $\mathbf{v}(0) = v_{x0}\hat{\mathbf{x}}$.

(1) Equations of motion

The Lorentz-force equation is

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad \mathbf{v} \times \mathbf{B} = Bv_y\hat{\mathbf{x}} - Bv_x\hat{\mathbf{y}}.$$

Therefore

$$\dot{v}_x = \frac{qB}{m}v_y, \quad \dot{v}_y = \frac{q}{m}(E - Bv_x), \quad \dot{v}_z = 0.$$

As $z(0) = v_z(0) = 0$, the particle remains in the xy plane.

(2) No-deflection velocity

Setting the transverse force to zero gives $E - Bv_x = 0$. Therefore

$$\boxed{\mathbf{v}_{\text{dr}} = \frac{E}{B}\hat{\mathbf{x}},}$$

with magnitude E/B in the $+x$ direction. This is the velocity selected by the crossed fields.

(3) Velocity for arbitrary v_{x0}

Set

$$\omega = \frac{qB}{m}, \quad v_{\text{dr}} = \frac{E}{B}, \quad u_x = v_x - v_{\text{dr}}.$$

Differentiate the first equation and substitute the second:

$$\ddot{u}_x = \omega\dot{v}_y = -\omega^2 u_x.$$

The general solution of this oscillator equation is

$$u_x = A \cos \omega t + C \sin \omega t.$$

Since $\dot{u}_x = \omega v_y$, differentiate this expression to obtain

$$v_y = -A \sin \omega t + C \cos \omega t.$$

At $t = 0$, $u_x(0) = u_0$ gives $A = u_0$, while $v_y(0) = 0$ gives $C = 0$. Therefore

$$\boxed{v_x(t) = v_{\text{dr}} + u_0 \cos \omega t, \quad v_y(t) = -u_0 \sin \omega t.}$$

(4) Position and trajectory types

Integrate each velocity component from 0 to t , using $x(0) = y(0) = 0$:

$$x(t) = \int_0^t (v_{\text{dr}} + u_0 \cos \omega \tau) d\tau, \quad y(t) = \int_0^t -u_0 \sin \omega \tau d\tau.$$

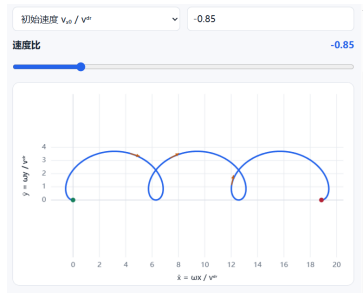
The antiderivatives are $\sin(\omega\tau)/\omega$ and $\cos(\omega\tau)/\omega$. Evaluating them between $\tau = 0$ and $\tau = t$ gives

$$x(t) = v_{\text{dr}}t + \frac{u_0}{\omega} \sin \omega t, \quad y(t) = \frac{u_0}{\omega} (\cos \omega t - 1).$$

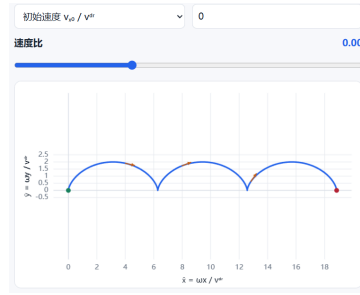
The periodic motion is superimposed on a mean drift toward $+x$. A loop occurs precisely when $v_x = v_{\text{dr}} + u_0 \cos \omega t$ becomes negative during part of a cycle.

- $v_{x0} = v_{\text{dr}}$: $u_0 = 0$, giving straight-line drift.
- $0 < v_{x0} < 2v_{\text{dr}}$, excluding v_{dr} : smooth curve without loops.
- $v_{x0} = 0$ or $v_{x0} = 2v_{\text{dr}}$: the velocity vanishes at some instants, giving cusps.
- $v_{x0} < 0$ or $v_{x0} > 2v_{\text{dr}}$: v_x changes sign and the curve has loops.

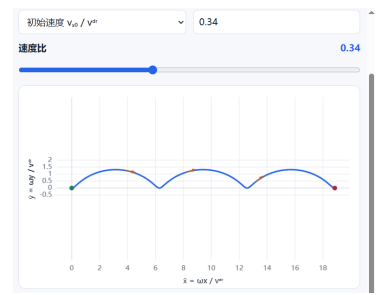
If $v_{x0} < 0$, the particle initially moves toward $-x$, though the average drift remains toward $+x$.



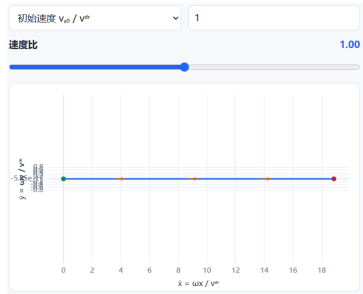
(a) $v_{x0} < 0$, loops



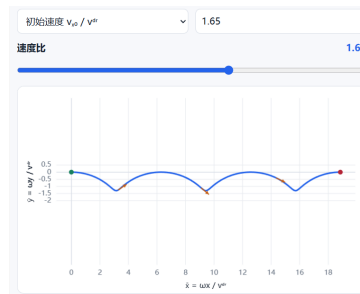
(b) $v_{x0} = 0$, cusps



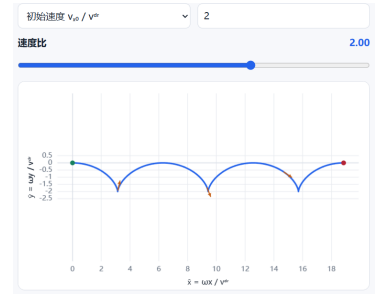
(c) $0 < v_{x0} < v_{\text{dr}}$, smooth



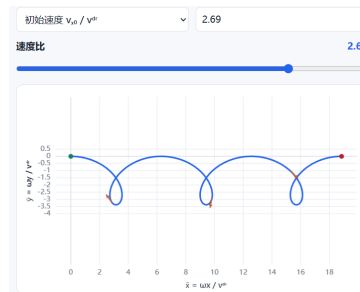
(d) $v_{x0} = v_{\text{dr}}$, straight line



(e) $v_{\text{dr}} < v_{x0} < 2v_{\text{dr}}$, smooth



(f) $v_{x0} = 2v_{\text{dr}}$, cusps



(g) $v_{x0} > 2v_{\text{dr}}$, loops

Figure 1

Board sketch. Show a line at $v_{x0} = v_{\text{dr}}$, a non-loop curve for $0 < v_{x0} < 2v_{\text{dr}}$, cusps at 0 and $2v_{\text{dr}}$, and loops outside that interval. Add arrows for increasing time.

3 Problem 3: The region accessible to a projectile

A projectile is launched from the origin at fixed speed v_0 . Gravity is $-g\hat{z}$; air resistance, obstacles, and ground are ignored. We seek the accessible region in cylindrical coordinates (ρ, ϕ, z) .

Strategy

For a target at horizontal distance $\rho > 0$, aim the horizontal velocity directly toward it. Let θ be the launch elevation angle from the horizontal. Resolve the initial velocity into horizontal and vertical components, find the height at horizontal distance ρ , and maximize over θ . Then show that the height function attains every value below its maximum.

Height at fixed $\rho > 0$

The launch components are

$$v_{h0} = v_0 \cos \theta, \quad v_{z0} = v_0 \sin \theta.$$

To reach positive horizontal distance, $\cos \theta > 0$, and the travel time is

$$t = \frac{\rho}{v_0 \cos \theta}.$$

The height at that time is

$$z(\theta) = v_0 \sin \theta t - \frac{1}{2}gt^2 = \rho \tan \theta - \frac{g\rho^2}{2v_0^2 \cos^2 \theta} = \rho \tan \theta - \frac{g\rho^2}{2v_0^2} (1 + \tan^2 \theta).$$

Differentiate directly with respect to $\tan \theta$:

$$\frac{dz}{d(\tan \theta)} = \rho - \frac{g\rho^2}{v_0^2} \tan \theta.$$

The maximum occurs when $\tan \theta_* = v_0^2/(g\rho)$. Substitution gives

$$z_{\max} = \rho \frac{v_0^2}{g\rho} - \frac{g\rho^2}{2v_0^2} \left(1 + \frac{v_0^4}{g^2 \rho^2} \right) = \frac{v_0^2}{g} - \frac{g\rho^2}{2v_0^2} - \frac{v_0^2}{2g} = \boxed{\frac{v_0^2}{2g} - \frac{g\rho^2}{2v_0^2}}.$$

Necessity and sufficiency

As θ ranges over $(-\pi/2, \pi/2)$, $\tan \theta$ ranges over all real numbers. The expression for z is a downward-opening quadratic in $\tan \theta$, so its range is $(-\infty, z_{\max}]$. Thus points above the boundary are inaccessible, and every point on or below it can be reached.

A horizontal velocity component perpendicular to the target direction cannot help reach that target and consumes part of the fixed speed. Directing all horizontal velocity toward the target therefore gives the greatest possible height at that ρ .

Axis case $\rho = 0$

Launch vertically. The maximum height is $v_0^2/(2g)$. Upward flight reaches every height between zero and this maximum. Since the problem ignores the ground, a downward launch reaches every negative height. This agrees with setting $\rho = 0$ in the boundary formula.

The accessible region and its boundary are

$$z \leq \frac{v_0^2}{2g} - \frac{g\rho^2}{2v_0^2}, \quad z_{limit} = \frac{v_0^2}{2g} - \frac{g\rho^2}{2v_0^2},$$

a paraboloid of revolution.

Teaching check. Prove both the upper bound and attainability below it. Treat $\rho = 0$ separately because the travel-time expression assumes positive horizontal distance.

4 Problem 4: Slowing down after a parachute opens

A payload and parachute have mass $m = 80 \text{ kg}$ and descend initially at $v_0 = 30 \text{ m s}^{-1}$. Take downward as positive. The drag magnitude is cv^2 , with $c = 8.0 \text{ kg m}^{-1}$. Use $g = 9.81 \text{ m s}^{-2}$.

(1) Equation and terminal speed

With downward positive,

$$m \frac{dv}{dt} = mg - cv^2.$$

At terminal speed $mg = cv_T^2$, so

$$v_T = \sqrt{\frac{mg}{c}} = \sqrt{98.1} \approx 9.90 \text{ m s}^{-1}.$$

Because $v_0 > v_T$, the initial acceleration is

$$\dot{v}(0) = g - \frac{c}{m}v_0^2 = 9.81 - 90 = -80.19 \text{ m s}^{-2}.$$

This acceleration is opposite to the downward velocity, so the payload initially slows.

(2) Speed and time to reach v_1

Using $v_T^2 = mg/c$,

$$\frac{dv}{dt} = g \left(1 - \frac{v^2}{v_T^2} \right).$$

For the branch $v > v_T$, separate variables:

$$\frac{dv}{1 - v^2/v_T^2} = g dt.$$

Use the partial-fraction identity

$$\frac{1}{1 - v^2/v_T^2} = \frac{v_T^2}{(v_T - v)(v_T + v)} = \frac{v_T}{2} \left(\frac{1}{v_T - v} + \frac{1}{v_T + v} \right).$$

For $v > v_T$, an antiderivative is

$$\int \frac{v_T}{2} \left(\frac{1}{v_T - v} + \frac{1}{v_T + v} \right) dv = \frac{v_T}{2} \ln \left(\frac{v + v_T}{v - v_T} \right).$$

Applying the limits v_0 at $t = 0$ and $v(t)$ at time t gives

$$\frac{v_T}{2} \left[\ln \left(\frac{v + v_T}{v - v_T} \right) \right]_{v_0}^{v(t)} = gt.$$

Equivalently, the velocity is specified by the logarithmic relation

$$\boxed{\ln \left(\frac{v(t) + v_T}{v(t) - v_T} \right) = \ln \left(\frac{v_0 + v_T}{v_0 - v_T} \right) + \frac{2g}{v_T} t.}$$

Setting $v(t_1) = v_1$ in the logarithmic solution and isolating t_1 gives

$$t_1 = \frac{v_T}{2g} \left[\ln \left(\frac{v_1 + v_T}{v_1 - v_T} \right) - \ln \left(\frac{v_0 + v_T}{v_0 - v_T} \right) \right] = \frac{v_T}{2g} \ln \left(\frac{(v_1 + v_T)(v_0 - v_T)}{(v_1 - v_T)(v_0 + v_T)} \right) \approx \boxed{0.838 \text{ s}}.$$

Here $v_1 > v_T$, so v_1 is reached in finite time; v_T is approached asymptotically.

(3) Distance descended

By the chain rule, $dv/dt = v dv/ds$, so

$$mv \frac{dv}{ds} = mg - cv^2.$$

As speed decreases while distance increases,

$$\Delta s = \int_{v_1}^{v_0} \frac{mv}{c(v^2 - v_T^2)} dv = \frac{m}{2c} [\ln(v^2 - v_T^2)]_{v_1}^{v_0} = \frac{m}{2c} \ln \left(\frac{v_0^2 - v_T^2}{v_1^2 - v_T^2} \right) \approx \boxed{14.3 \text{ m}}.$$

Teaching check. The drag term is negative for downward-positive coordinates. Since $v_0 > v_T$, use the logarithmic branch with $v(t) > v_T$ and reverse the speed integration limits to obtain positive distance.

5 Problem 5: The longest fall

Earth is a nonrotating sphere of radius R , mass M , and uniform density. Tunnels are evacuated and friction is ignored.

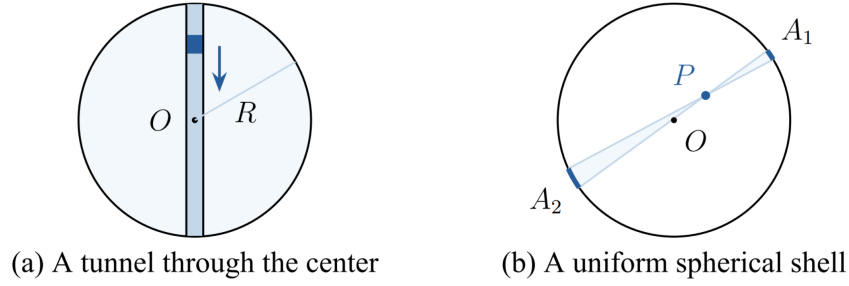


Figure 2

(1) Field inside a thin spherical shell

Choose an interior point P . Opposite narrow cones with vertex P subtend equal infinitesimal solid angles $d\Omega$ and cut patches A_1, A_2 . Let the shell surface density be σ . For a patch at distance r , whose normal makes angle α with the line of sight,

$$dA = \frac{r^2 d\Omega}{|\cos \alpha|}, \quad dF = \frac{G\sigma dA}{r^2} = \frac{G\sigma d\Omega}{|\cos \alpha|}.$$

The inverse-square factor cancels r^2 , but the inclination correction remains. Let $d = OP$ and use OP as the axis. The paired directions make angles θ and $\pi - \theta$ with it. For shell radius R , sphere geometry gives the same normal projection factor for both patches:

$$|\cos \alpha| = \frac{\sqrt{R^2 - d^2 \sin^2 \theta}}{R}.$$

Their force magnitudes are equal and their axial components are opposite; rotational symmetry cancels transverse components. Hence

$$\boxed{\mathbf{g}_{\text{shell}}(P) = \mathbf{0}.$$

(2) Diameter tunnel

The mass enclosed within radius $r \leq R$ is

$$M(r) = M \frac{r^3}{R^3}.$$

Outer shells contribute no net field, so the inward field magnitude is

$$g(r) = \frac{GM(r)}{r^2} = \frac{GM}{R^3} r.$$

For signed coordinate x along a diameter,

$$\ddot{x} = -\frac{GM}{R^3} x.$$

This is simple harmonic motion, with

$$\omega = \sqrt{\frac{GM}{R^3}}, \quad \boxed{T = 2\pi \sqrt{\frac{R^3}{GM}}}.$$

The general solution is $x(t) = A \cos \omega t + B \sin \omega t$. From $x(0) = R$, $A = R$; from $\dot{x}(0) = 0$, $B = 0$. Thus $x(t) = R \cos \omega t$. The opposite end is reached in

$$\boxed{\frac{T}{2} = \pi \sqrt{\frac{R^3}{GM}}}.$$

(3) Circular-orbit projection

For a circular orbit of radius approximately R ,

$$\frac{GMm}{R^2} = mR\omega_{\text{orb}}^2, \quad \omega_{\text{orb}} = \sqrt{\frac{GM}{R^3}} = \omega.$$

A coordinate of this circular motion is $R \cos(\omega t)$ for a suitable initial phase, identical to the tunnel coordinate. The tunnel motion is the projection of the circular motion onto the tunnel axis.

(4) Arbitrary chord tunnel

Let $\hat{\mathbf{e}}$ point along the chord, let $\mathbf{b}$ be the perpendicular vector from Earth's center to the chord midpoint, and let ξ measure position from the midpoint. Then

$$\mathbf{r} = \mathbf{b} + \xi \hat{\mathbf{e}}, \quad \mathbf{b} \cdot \hat{\mathbf{e}} = 0.$$

Inside the uniform sphere, $\ddot{\mathbf{r}} = -(GM/R^3)\mathbf{r}$. Projecting along the tunnel,

$$\ddot{\xi} = -\frac{GM}{R^3}(\mathbf{b} + \xi \hat{\mathbf{e}}) \cdot \hat{\mathbf{e}} = -\frac{GM}{R^3}\xi.$$

It is simple harmonic motion about the chord midpoint, with

$$\boxed{T = 2\pi \sqrt{\frac{R^3}{GM}}}.$$

The period is independent of the chord's distance from Earth's center. The offset changes the chord length and amplitude, not the restoring-force coefficient. Travel from one end to the other takes $T/2$.