# Lecture 9: Conservation Law One: Energy Conservation (part II)

**Several forces:**

If all of them are conservative forces, each of them gives rise to a potential:

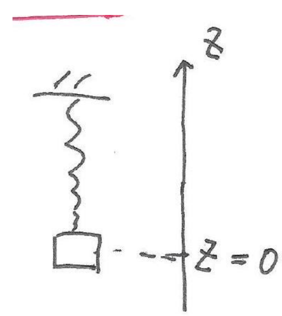
$$\vec{F}_1 = -\nabla U_1, \vec{F}_2 = -\nabla U_2, ..$$

then

$$E = E_k + U_1(\vec{r}) + U_2(\vec{r}) + \cdots \quad \text{is conserved}$$

example: spring in a gravity field

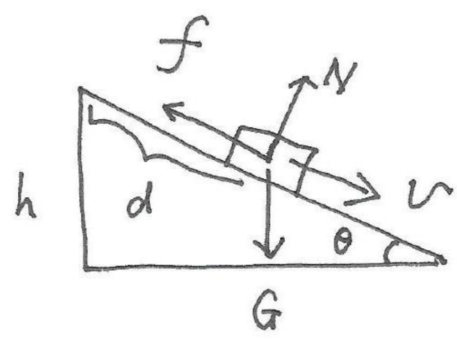
$$E = E_k + mgz + \frac{1}{2}kz^2$$




**Non-conservative force**

$$\Delta E_k = W_{\text{con}} + W_{\text{ncon}} \ = -\Delta U + W_{\text{ncon}}$$
$$\Rightarrow \Delta\left(E_k + U\right) = W_{\text{ncon}}$$

example: $\Delta\left(E_k + U\right) = -fd = mg\sin\theta\mu d$



$E_{k,i} = 0, \quad U_i = mgh = mgd\sin\theta$

$T_f = ? \quad U_f = 0 \quad \Rightarrow T_f = mgd\sin\theta - mgd\cos\theta\mu = \frac{1}{2}mv_f^2$

$v_f = \sqrt{2gd(\sin\theta - \mu\cos\theta)}$

**1D motion:** $F_x$

If $F_x$ is only coordinate-dependent, then $F_x$ is conservative. This is because any closed loop in 1D has to ame back along the same path $\int_1^2 dxF_x + \int_2^1 dxF_x = 0$


Then the potential energy $U(x)$ can be simply integrated as

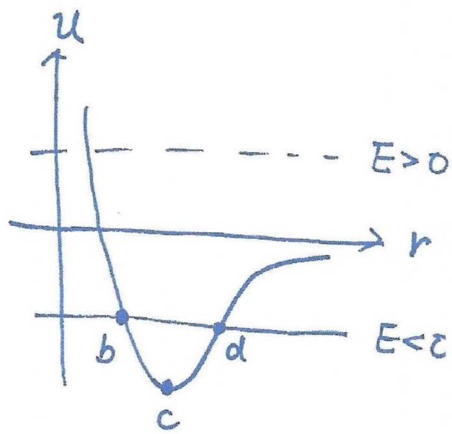
$$U(x) = -\int_{x_0}^{x} F_x(x')\,dx'$$

$x_0$ can be any point $U(x)$ with different $x_0$ is up to a constant.

**example:** potential energy for a diatomic molecule
(1) $E < 0$ - bound states
At $b$ and $d \Rightarrow T = 0$, turning points. At c,

$$\frac{\partial U}{\partial r} = 0, \frac{\partial^2 U}{\partial r^2} > 0.$$

$c$ is equilibrium point




(2) $E > 0$ - scattering states
We can formally complete the solution of motion in 1D

$$T = \frac{1}{2}m\dot{x}^2 = E - U(x) \Rightarrow \dot{x}(x) = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$

The direction of $\dot{x}(x)$ can be either right/left mover.
we also have

$$\dot{x}=\frac{dx}{dt}\Rightarrow dt=\frac{dx}{\dot{x}(x)}\Rightarrow \quad \int_{t_i}^{t_f}dt=\int_{x_i}^{x_f}\frac{dx}{\dot{x}(x)}=t_f-t_i$$

Suppose $\dot{x}$ is positive, we have

$$t_f-t_i=\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E-u\left(x'\right)}}.$$

$\dot{x}$ can change directions at turning points, and we can treat by dividing the motion into different regions. In each region, $\dot{x}$ 's direction is fixed, and we add the time of each region together.

**Example:**
1) free fall:

$$U\left('z\right)=-mgz$$

and

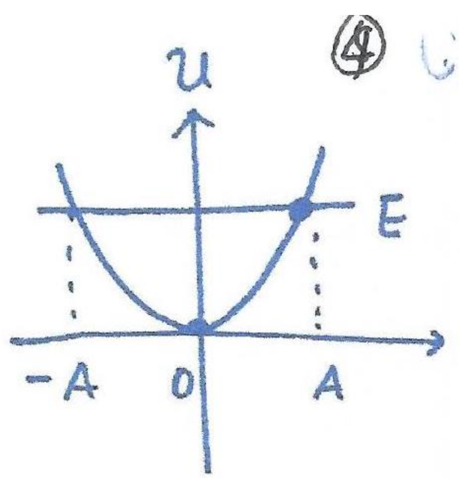
$$\Rightarrow \dot{z}(z)=\sqrt{\frac{2}{m}}\sqrt{E-U(z)}=\sqrt{2gz}$$

E=0, $v_{in}=0$ at z=0.

$$t=\int_0^z\frac{dz'}{\dot{z}\left(z'\right)}=\int_0^z\frac{dz'}{\sqrt{2gz'}}=\sqrt{\frac{2z}{g}}\Rightarrow z=\frac{1}{2}gt^2$$

2) harmonic oscillator:
$U=\frac{1}{2}kx^2$ with energy $E$.
The turning points at $\pm A$, with

$$\frac{1}{2}kA^2=E.$$




consider at $\begin{cases} t_{\text{in}} &= 0 \\ x_0 = A \end{cases}$ and at $\begin{cases} t_f = T/4 \\ x_f = 0 \end{cases}$

we have

$$\dot{x}(x) = -\sqrt{\frac{2}{m}}\left(E - \frac{1}{2}kx^2\right)^{1/2}$$

$$\Rightarrow T/4 = +\int_A^0 \frac{dx}{\dot{x}} = \sqrt{\frac{m}{2}}\int_0^A dx \frac{1}{\left(E - \frac{1}{2}kx^2\right)^{1/2}}$$

$$= \sqrt{\frac{m}{2}}\left(\frac{k}{2}\right)^{-1/2} \cdot \int_0^A dx \frac{1}{A\left(1 - \left(\frac{x}{A}\right)^2\right)^{1/2}}$$

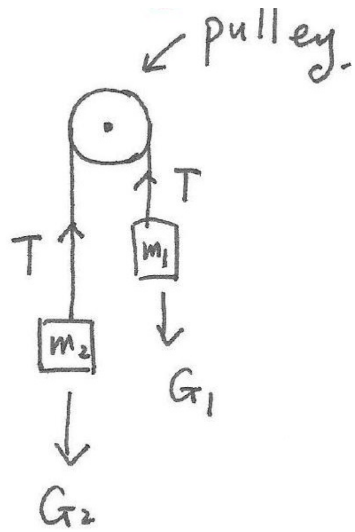
$$= \quad \sqrt{\frac{m}{k}}\int_0^1 dy \frac{1}{\left(1 - y^2\right)^{1/2}} = \omega_0^{-1} \arcsin y\big|_0^1 = \frac{\pi}{2\omega_0}$$

$$\Rightarrow T = \frac{2\pi}{\omega_0} \quad \text{where } \omega_0 = \sqrt{k/m}.$$

**Several objects: Atwood machine with constraints**

Two masses suspended by a massless $s$ inextensible string

$$\Delta\left(E_{k_1} + u_1\right) = W_1^T$$
$$\Delta\left(E_{k_2} + u_2\right) = W_2^T$$




Tensions on $m_1$ and $m_2$ are the same, but $d\left(S_1 + S_2\right) = 0 \rightarrow$ constraint

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No elasticity, hence

$$W_1^{\text{ten}} + W_2^{\text{ten}} = \int ds_1 W_1^{\text{ten}} + \int ds_2 W_2^{\text{ten}} = \int ds_1 T + \int ds_2 T$$
$$= \int T\left(ds_1 + ds_2\right) = 0$$
$$\Rightarrow \Delta\left(E_{k_1} + E_{k_2} + u_1 + u_2\right) = 0$$

**In general**, if a system contains several particles, with constraints, and if the constraining force does not do the total work on the system, they can be neglected in the total energy.

**Energy of two interacting particles**

$$\begin{cases} \vec{F}_{12} = \vec{F}_{12}\left(\vec{r}_1 - \vec{r}_2\right) & - \text{ translation symmetry} \\ \vec{F}_{12} = -\vec{F}_{21} & \text{- interaction only depends on the relative displacement} \end{cases}$$


$\vec{F}_{12}\left(\vec{r}_1 - \vec{r}_2\right)$, if for fixed $\vec{r}_2$, is a conservative force for $\vec{r}_1$, i.e. $\oint d\vec{r}_1 \cdot \vec{F}_{12}\left(\vec{r}_1 - \vec{r}_2\right) = 0$, then we express

$$\vec{F}_{12} = -\nabla_{\vec{r}_1} U\left(\vec{r}_1 - \vec{r}_2\right).$$

then the same potential can also give rise to

$$\vec{F}_{21} = -\nabla_{\vec{r}_2} U_{12}\left(\vec{r}_1 - \vec{r}_2\right) = -\vec{F}_{12} \rightarrow \text{Newton's 3rd law}$$

Now apply the work-kinetic theorem,

$$\left.\begin{array}{l} dE_{k_1} = d\vec{r}_1 \cdot \vec{r}_{12} \\ dE_{k_2} = d\vec{r}_2 \cdot \vec{r}_{21} \end{array}\right\} \Rightarrow$$

$$d\left(E_{k_1} + E_{k_2}\right) = \vec{F}_{12} \cdot \left(d\vec{r}_1 - d\vec{r}_2\right)$$
$$= d\left(\vec{r}_1 - \vec{r}_2\right) \cdot \left(-\nabla_1 U_{12}\left(\vec{r}_1 - \vec{r}_2\right)\right)$$
$$= -d\vec{r} \cdot \nabla u_{12}(\vec{r}) = -dU(\vec{r})$$

$\vec{r} = \vec{r}_1 - \vec{r}_2 \leftarrow$ relative coordinate

$$d(\underbrace{E_{k_1} + E_{k_2} + U(\vec{r})}_{E}) = 0$$

In principle, we can also include the external conservative forces on 1 and 2, and introduce potentials. $U_1^{\text{ex}}$ and $U_2^{\text{ex}}$, then

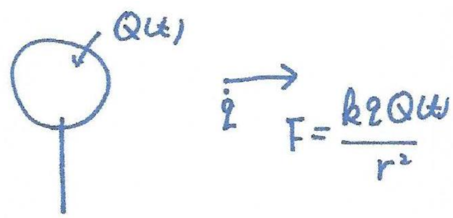
$$E = T_1 + T_2 + u_1^{ey_s} + u_2^{ex} + u_{12}.$$

This process can be generalized to $n$-particle conservative systems. With

$$\begin{aligned} E &= T_1 + T_2 + \cdots T_n + U_1^{ex} + U_2^{ex} + \cdots U_n^{ex} \\ &\quad + U_{12} + \cdots U_{1n} + U_{23} + \cdots U_{2n} + \cdots U_{n-1,n} \\ E &= \sum_{i=1}^{n} (T_i + U_i^{ex}) \leftarrow \text{ single body} \\ &\quad + \sum_{i<j} U_{ij} \leftarrow \text{ interaction (double-counting excluded)} \end{aligned}$$

**Time-dependent potential energy**

If $\vec{F}(\vec{r}, t)$ satisfies $\oint d\vec{r} \cdot \vec{F}(\vec{r}, t) = 0$, but it's time-dependent, then we can still write $\vec{F}(\vec{r}, t) = -\nabla U(\vec{r}, t)$. Nevertheless $E = T + U$ is no longer conserved.




For a changing charge $Q(t)$, we can still define

$$U(\vec{r}, t) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}', t)\, d\vec{r}'.$$

Now check

$$dT = \frac{dT}{dt} dt = \frac{d}{dt}\left(\frac{1}{2} m v^2\right) dt = m\dot{\vec{v}} \cdot \vec{v} dt = \vec{F} \cdot d\vec{r}$$

$$\begin{aligned} dU(\vec{r}, t) &= \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz + \frac{\partial U}{\partial t} dt \\ &= \nabla U \cdot d\vec{r} + \frac{\partial U}{\partial t} dt = -\vec{F} \cdot d\vec{r} + \frac{\partial U}{\partial t} dt \\ &\Rightarrow dT = -dU + \frac{\partial U}{\partial t} dt \quad \Rightarrow d(T + U) = \frac{\partial U}{\partial t} dt \end{aligned}$$