Lecture 8: Conservation Law One: Energy Conservation (part I)

In principle, once we have had Newton's law of motion, it should be able to determine the motion of equation by solving the 2nd order differential equations and substituting the initial conditions. Nevertheless, the perspective of conservation laws is a major development of Newtonian mechanics whose importance is at least two-fold. One side is empirical: The 2nd order differential equations arising from Newton's 2nd law are often difficult to solve. If we can find a constant of motion, the 2nd order differential equations can often be reduced to the first order one, whose solution would be much easier. The other side is more profound, conservation laws reflect deeply the symmetry structures of a mechanical system, which are consequences of the nature of fundamental space-time. We will learn three types of conservation laws:, energy conservation, momentum conservation, and angular momentum conservation, which are consequences of time homogeneity, spatial homogeneity, and spatial isotropy, respectively.

8.1 "Dead force" and "living force (vis viva)"

Although the universe is dynamic, it would be interesting to find something behind the dynamic phenomena that is invariant. The ideology of "conservation" even appeared in the ancient Greece time. Thales of Miletus (about 550 BC) proposed that the basic substance that everything is made of, which he felt should be water, is conserved. Galileo pointed out that a body moving along a smooth curve falls from a certain height, and finally can rise to an equal height independent of the concrete shape of the curve, which today is understood as the conservation of mechanical energy.

Descartes (1596-1650) thought that the total quantity of motion in the universe must be a constant. Since moving objects change their velocities by collisions, he studied collisions to seek the conservation quantities in motion. He proposed to use the product of mass and speed, i.e., mv, which is later on "momentum", as the measure of the quantity of motion. Descartes did not figure out the vector nature of momentum, nevertheless, his research pointed out a long term direction to seek conservation laws. On the other hand, Huygens (1629-1695) proposed a new conserved quantity, the sum of the product of each mass and its velocity square, which is equivalent to the kinetic energy in today's language. Leibniz (1646-1716) named this quantity as "vis viva", or, living force, and he called Descartes' mv as "dead force".

Leibniz pointed out if an object with the weight mg is raised to a height of h, as long as mgh is the same, the object acquires the same amount of vis viva. Consider that a falling object smashes a nail: An object with the weight of mg falling from the height $4h$ and an object with the weight of $4m$ falling from the height h smash the nail to the same depth. Their living forces are the same, i.e., $2mgh$, but their dead force are different, the heavy object is $m\sqrt{32gh}$, and the light one is $m\sqrt{8gh}$. The former is twice larger than the latter.

This dispute that whether the "dead force" or "living force" is more essential was settled down by d'Alembert (1717-1783). He recognized the importance of both, which are quantities measuring different aspects of motion. Again consider the above example, two falling objects with the same living force but different dead force. The heavy object has twice amount of the dead force than the latter case. Although they smash the nail to the same depth, but the periods of time they spent are different. The light object falls from a higher place, hence, its speed is higher, then during its collision with the nail, less time is taken to smash the nail to the same depth. Similarly, the heavy object has larger "dead force", which needs more time to stop, consistent with its lower speed during the smashing or collision process.

The perspective of the conservation of vis viva was promoted by Johann Bernoulli and Daniel Bernoulli. The father developed the virtual work principle in statics, and the son developed Bernoulli's principle for the change of the hydrodynamic pressure. Further developments led to the appearance of Analytic Mechanics, including the Lagrangian, and Hamiltonian formalisms, which you will learn in the course of "Classical Mechanics".

In the early 19th century, vis viva was recalibrated by Coriolis and Poncelet et. al. to the modern form of

$$\frac{1}{2}\sum_i m_i v_i^2 \tag{7.1}$$

which can be completely used to do work, for example, to raise a weight to a certain height.

8.2 Gravitational potential energy - the reversible and irreversible weight-lifting machines

We have the impression that if a weight G is lifted to a height h, it can be used to do work, hence, it intuitively has a certain type of energy. Later we will denote it as the gravitational potential energy. Then how to define it? The definition has to be universal, i.e., independent of concrete material of the object and how it is lifted, etc.

We must provide a starting point for reasoning, which is similar to the postulate of Euclidean geometry, which is simply Postulate: No perpetual machine exists.

The study of "perpetual machine" has a long history. The failure of construction of the perpetual machine is significant, which leads to the discovery of the law of energy conservation. The machines we are talking now are only weight-lifting machines, and later on when learning thermodynamics we will talk on thermal engines. In our context, "no perpetual machine" meanings that it is impossible to have a net result of lifting a weight after the machine and everything in the environment are restored to their initial states. In other words, no free lunch and no pains no gains. You must pay efforts to lift a weight.

A weight-lifting machine is sketched in Fig. 7.1: A mass of m is hung vertically with a rope passing a pulley connecting to another mass M on a slope. m and the slope is viewed as the weight-lift machine, and M is the object to lift. If m drops at a distance of h, then M is lifted at the height h' which is determined by the slope angle θ, i.e.,

$$h'/h = \sin\theta. \tag{7.2}$$

But for different kinds of slopes, apparently, the value of M which can be lifted for the same of m is different.

We define "reversible" and "irreversible" weight-lifting machines. Reversible machines means that after the machine is operated, the process can be reversed to restore the system to initial conditions without any net changes. For example, if the slope is slippery, i.e., no friction, it is reversible. According to its geometric shape, we prepare the mass M such than it balances m. The weightlifting ratio R is defined as

$$R = M/m. \tag{7.3}$$

We push m downward a little bit to let it just move at an infinitesimal speed. The process can be reversed: we push M downward a little bit to make it move back and then m rises back to its initial position. Then we give M a tiny push

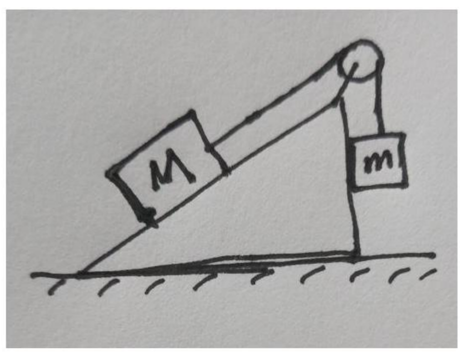


Figure 8.1 A slope as weight-lifting machine. No net weight should be lifted if this machine and its environment are restored to the initial conditions.

to stop the motion. These pushes can be made as small as possible, hence, they can be neglected if we are gentle enough.

On the other hand, if frictions exist, the process becomes irreversible. In order for m to move downward, M is expected to be lighter than before. In this case, the weight-lifting ratio is defined as $R = M/m$ when m drops and pulls up M in a uniform speed, i.e., in a balanced way. In this case, it is impossible to reverse the motion by a tiny push. An extra force is needed to push M downward since the friction switches the direction. In order to restore the machine to the initial condition, extra work is done which finally is converted to heat by friction.

A natural question arise: for a given slope angle, what kind of machines have the highest ratio of R, i.e., the largest M that a mass m can raise? The answer is the reversible ones.

Consider two weight-lifting machines: one irreversible and one reversible whose ratios of M/m are R_i and R_r, respectively. First, run the irreversible one, i.e., drop m at a distance of h, then $M_i = mR_i$ is raised to the height of h'. Then two masses are shifted to the slippery slope without friction without motion in the vertical direction (Suppose that this could be done with care and patience). If $R_i > R_r$, then M_i is more than enough to go down and pull m up. Part of its mass $\Delta M = (R_i - R_r)\,m$ is cut and is held at its position, and the rest part is reduced to $M_r = R_r m$, which is just right to run the reversible machine in the reverse way. Then m is restored to its original height, M_r also goes back to its location, but an amount of ΔM is lifted for a height of h'. Then according to our postulate, this is impossible. Hence $R_i \leqslant R_r$, i.e., for all the weight-lifting machines, the reversible one can lift the largest weight.

How about two reversible machines? Since both of them are reversible, we

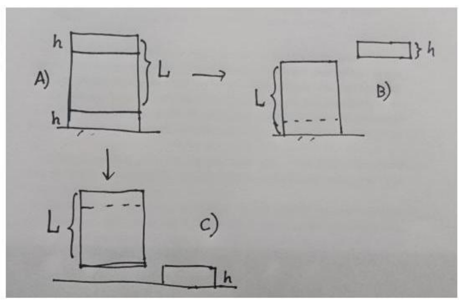


Figure 8.2 A) An object with the height of $L + h$ is cut two pieces from the top and the bottom at the thickness of h. B) The top piece is shifted to right. C) The bottom piece is shifted to right. These three configurations have the same amount of gravitational potential energy.

can reverse either of them and perform the above reasoning. Then the ratios of $R's$ of these two reversible machines have to be equal to each other.

We have used a concrete design of weight-lifting machine - the slope. Actually, if you examine the above reasoning, the following conclusion is very general regardless of the concrete design: if a mass m is dropped at a vertical distance h, if the lifting distance is h', then the reversible machines lift the largest weight. For all the reversible weight-lifting machines, the mass lifted M should be same.

Then what is the value of the largest mass M? Since the design of the liftweighting machine is unimportant, we consider another one depicted in Fig. 7.2. Consider a cylindrical object with a uniform density and a height $L + h$. Imagine we cut two pieces with the thickness of h from the top and the bottom, respectively. The top piece can be shifted to the right and is held at the same height as depicted in Fig. 7.3 (b). Similarly, the bottom piece is shifted, and the rest major part is held at its original position as depicted in Fig. 7.3 (c). Now we compare Fig. 7.3 (b) and (c), since both of them have no vertical motion compared to Fig. 7.3 (a), they can be viewed as a small object with mass m is lowered by a distance L which lift the bigger object with mass M at a distance of h. Since the ratio of m/M is also h/L,

$$mgL + Mg(-h) = 0, \tag{7.4}$$

where the gravity acceleration g is multiplied since the system is in the gravity field.

Then we conclude that for a reversible weight-lifting machine (no friction), if its different parts are moving in an infinitesimally slow way, i.e., adiabatically, the product of the mass and the vertical distance lowered equals that

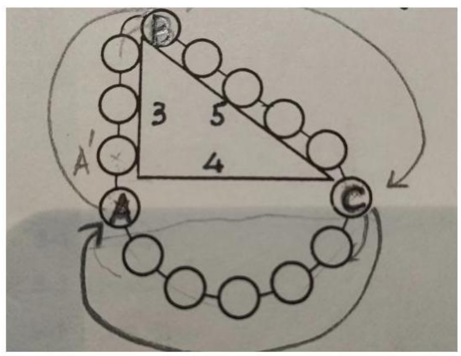


Figure 8.3 The inscription on Simon Stevin's (1548-1640) tombstone.

between the mass and the vertical distance lifted, hence, the following quantity, the gravitational potential is conserved

$$E_p = \sum_i m_i g z_i, \tag{7.5}$$

where z_i is the height in the gravity field and the ground is set as $z = 0$. In other words, the gravitational potential can be viewed as the sum of the force multiplied by the distance that the force acts through.

If we apply the potential energy conservation law to in the set up of Fig. 7.1, the ratio of $R = M/m = h/h' = 1/\sin\theta$. This results was already known by Simon Stevin (1548-1620). On his tombstone a picture is inscribed: A loop of balls are around the rectangular slope. Without of loss of generality, the shorter sides and the hypotenuse (the longest side) exhibit the ratio of $3 : 4 : 5$. The loop is at balance, hence the number ratio of balls on the vertical edge and the slope is $3 : 5$. It is easy to check that the balls hung below the slope are balanced by itself, hence, the three balls on the vertical edge balance the five balls on the slope.

8.3 Kinetic energy

We only considered infinitesimally slow motion above, hence, only the gravitational potential is needed. Nevertheless, physical intuition tells us that an object falling from a height will increase its speed. Now we will use the postulate of the impossibility of perpetual machine to study the speed of an object after falling a height.

Now we consider a new type of machines - "kinetic-energy machine" as

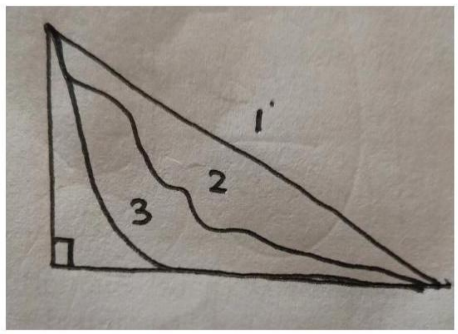


Figure 8.4 The "kinetic-energy" machines which convert the height to speed. For reversible machines, an object acquires the same speed after descending the same height.

depicted in Fig. 7.4: A ball falls at the same vertical distance but along different curves. Again we divide these machines into reversible and irreversible classes.

Reversible kinetic-energy machines mean that if the motion of the ball at the bottom is reversed but with the same speed, it will reach where it drops. Again following similar reasonings in the previous section, we conclude that the reversible machines exhibit the largest speeds, and all reversible machines exhibit the same speed. Otherwise, we could lift the ball without paying any effort.

Then we can use the straight-line slope to calculate the speed after it moves to the bottom. The force F to pull the ball with the mass M on the slope can be balanced by a mass $m = M \sin\theta$, hence,

$$a = F/M = mg/M = g\sin\theta. \tag{7.6}$$

Then

$$v^2 = 2al = 2gl\sin\theta = 2gh, \tag{7.7}$$

which is independent of the slope angle. We arrive at

$$\frac{1}{2}mv^2 = mgh. \tag{7.8}$$

The right-hand side is the gravitational energy drop, hence, if we interpret the left-hand side as the gain of another type of energy - the kinetic energy, then the total energy should be conserved. Then we define the kinetic energy as

$$E_K = \frac{1}{2}mv^2. \tag{7.9}$$

During this process, we have the conservation of mechanical energy in a reversible system which is defined as

$$E_M = E_K + E_p = \frac{1}{2}mv^2 + mgz \tag{7.10}$$

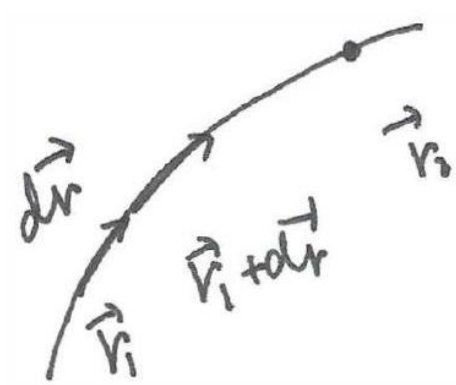
8.4 Work and energy

Now we come to concrete developments by using calculus, seeking conserved quantities, or constants of motion.

$$\begin{aligned}
\vec{F} = m\frac{d^2\vec{r}}{dt^2} &\Rightarrow \vec{F}\cdot d\vec{r} = md\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} \\
&= m\left[dx\cdot\frac{d^2x}{dt^2} + dy\cdot\frac{d^2y}{dt^2} + dz\cdot\frac{d^2z}{dt^2}\right] \\
\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = \frac{d}{dt}\left[\frac{dx}{dt}\cdot\frac{dx}{dt}\right]\cdot\frac{1}{2} &\Rightarrow \frac{d^2x}{dt}dx = \frac{1}{2}d\left[\left(\frac{dx}{dt}\right)^2\right] \\
\Rightarrow dE_k = \vec{F}\cdot d\vec{r}\ ^{\dagger},\ \text{where } E_k &= \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \\
&= \frac{1}{2}mv^2
\end{aligned}$$

†: work-energy theorem

linear integral

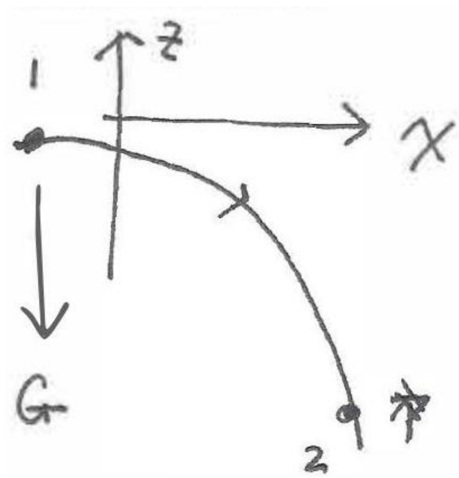


$$E_{k_2} - E_{k_1} = \sum \Delta\vec{r} \cdot \vec{F}$$
$$= \int_1^2 d\vec{r} \cdot \vec{F}$$

along the path from $1 \to 2$.

Example:
(1)projectile motion

$$F_{k_2} - E_{k_1} = \int_1^2 d\vec{r} \cdot \vec{G}$$

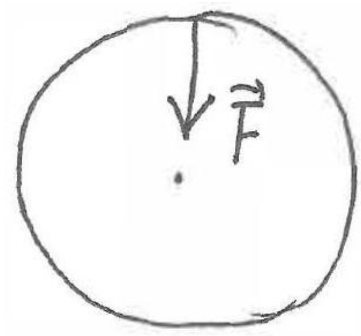


since

$$\vec{G} = -mg\hat{z}$$
$$\Rightarrow \Delta E_k = \int_{z_1}^{z_2} dz(-mg) = -mg\left(z_2 - z_1\right)$$
$$\Rightarrow \Delta E_k + \Delta(mgz) = 0, \text{ i.e } \Delta\left(E_k + mgz\right) = 0.$$

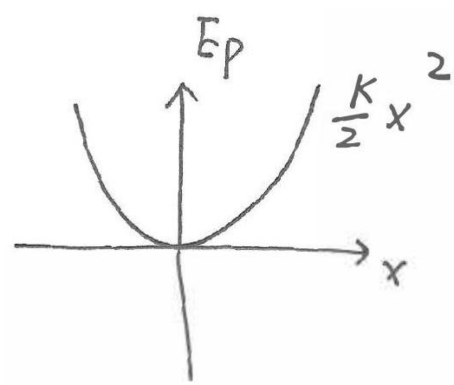
(2)Uniform speed circular motion

$$E_{k_2} - E_{k_1} = \int_1^2 d\vec{r} \cdot \vec{F} = 0$$
$$\Rightarrow \Delta E_k = 0$$



(3) Elastic restoring force

$$F = -kx$$
$$\Delta E_k = \int_{x_1}^{x_2} dx F \cdot x = -\frac{k}{2}\left(x_2^2 - x_1^2\right)$$
$$\Rightarrow \Delta\left(\frac{1}{2}mv^2 + \frac{k}{2}x^2\right) = 0$$
$$\Rightarrow \Delta(\text{kinetic energy} + \text{elastic energy}) = 0$$



Conservative forces:

For a general force $\vec{F}$, does $\int_1^2 d\vec{r} \cdot \vec{F}$ only depend on the initial and ending points (conservative), or does it depend on a concrete path (non-conservative)?
If $\vec{F}(\vec{r})$ is conservative, we can choose a reference point $\vec{r}_0$, and define

$$-u\left(\vec{r}_1\right) = -u\left(\vec{r}_0\right) + \int_{\vec{r}_0}^{\vec{r}_1} d\vec{r} \cdot \vec{F}(\vec{r})$$

The result is unique since the integral only depends on $\vec{r}_0$ and $\vec{r}_1$, but not on a concrete path. Then $u\left(\overrightarrow{r_1}\right) - u\left(\overrightarrow{r_0}\right) + E_{k_1} - E_{k_0} = 0$

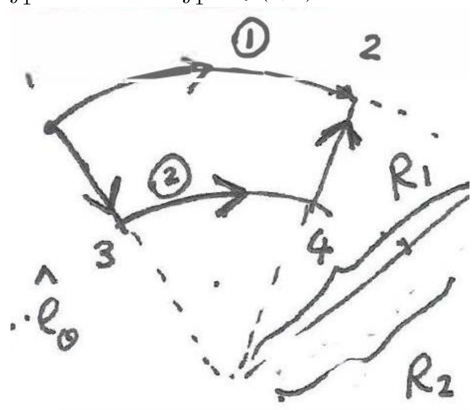
$$\Rightarrow \Delta\left(u(r)+E_k\right)=0$$

but not every force field satisfies this condition!

For example:

$$\vec{F}=f(r,\theta)\hat{e}_r$$

along path 1 $\Rightarrow \int_1^2 d\vec{r}\cdot\vec{F}=R_1\int_1^2 d\theta f(r,\theta)\hat{e}_r\cdot\hat{e}_\theta=0$



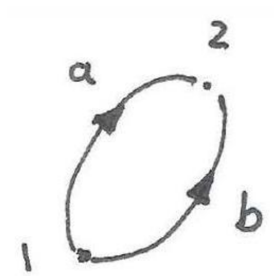
along path 2 $\Rightarrow \int_1^2 d\vec{r}\cdot\vec{F}=\left[\int_1^3+\int_3^4+\int_4^2 d\vec{r}\cdot\vec{F}\right]$

$$=-\int_{R_2}^{R_1} drf\left(r,\theta_{13}\right)+\int_{R_2}^{R_1} drf\left(r,\theta_{24}\right)$$

$$=\int_{R_2}^{R_1} dr\left(r,\theta_{24}\right)-f\left(r,\theta_{13}\right)\Big)$$

For those forces satisfying $\int_1^2 \vec{F}\cdot d\vec{r}$ independent on the paths, we denote them as conservative forces. Its equivalent expression is along any closed paths, the work done is zero.

$$\oint \vec{F}\cdot d\vec{r}=0$$



Proof: (1) If the force is conservative:
Consider a loop, we pick up two points 1,2 on the loop.

$$\Rightarrow \oint \vec{F} \cdot d\vec{r} = \int_1^2 d\vec{r} \cdot \vec{F} + \int_2^1 d\vec{r} \cdot \vec{F} = \int_1^2 d\vec{r} \cdot \vec{F} - \int_1^2 d\vec{r} \cdot \vec{F} = 0.$$

(2)If $\oint \vec{\phi} \cdot d\vec{r} = 0$, then:
Consider two paths **a** and **b** connecting positions 1 and 2 then

$$\int_{\text{path a}}^2 d\vec{r} \cdot \vec{F} - \int_{\text{path b}}^2 d\vec{r} \cdot \vec{F} = \int_1^2 d\vec{r} \cdot \vec{F} + \int_2^1 d\vec{r} \cdot \vec{F} = \oint d\vec{r} \cdot \vec{F} = 0$$

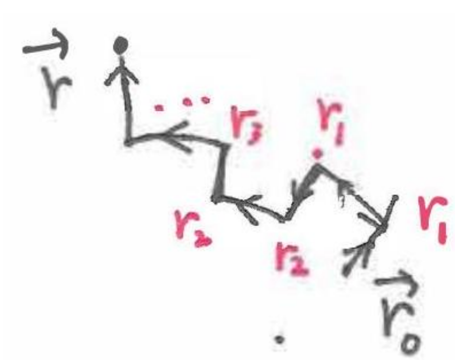
Hence, $\int d\vec{r} \cdot \vec{F}$ is path independent.

For a conservative force:

$$\begin{aligned} \vec{F} d\vec{r} &= F_x dx + F_y dy + F_z dz = -du \\ &= -\left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \right) \end{aligned}$$

$$\left.\begin{aligned} F_x &= -\tfrac{\partial u}{\partial x_1} \\ F_y &= -\tfrac{\partial u}{\partial y} \\ F_z &= -\tfrac{\partial u}{\partial z} \end{aligned}\right\} \Rightarrow \vec{F} = -\vec{\nabla} u \rightarrow \text{gradient}$$

Coulomb potential: $\vec{F}(\vec{r}) = \frac{qQ}{r^2} \hat{e}_r$
$\vec{u}\left(\vec{r}_1\right) - \vec{u}\left(\vec{r}_0\right) = -\int_{\vec{r}_0}^{\vec{r}_1} d\vec{r} \cdot \vec{F}(\vec{r})$



We can choose an arbitrary path, and decompose the path into the radial and angular directions. For those segments along the angular direction, $d\vec{r} \cdot \vec{F} = 0$, i.e., no work. We only need to calculate the integral along the radial direction,

$$\vec{u}(\vec{r}_1) - \vec{u}(\vec{r}_0) = -\left[\int_{r_0}^{r_1} drF(r) + \int_{r_1}^{r_2} drF(r) + \cdots \int_{r_n}^{r} drF(r)\right]$$
$$= -\int_{r_0}^{r_1} drF(r) = -qQ\int_{r_0}^{r_1} \frac{dr}{r^2} = qQ\left[\frac{1}{r}\right]\Big|_{r_0}^{r_1}$$

if we set $r_0 \to +\infty, u(r_0) = 0$, we arrive at $u(r) = \frac{qQ}{r}$.
The electric potential is $\frac{1}{q}u(r) = V(r) = \frac{Q}{r}$.
Similarly, we can define the gravitational potential on a large scale

$$u(r) = -\frac{GMm}{r}$$

Then the conservation of energy is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$