Lecture 7: Galilean transformation: inertial and non-inertial frames

Outline
(1) Inertial frames, Galilean relativity
All inertial frames are equal. One cannot distinguish one inertial frame from another.
(2) Non-inertial frames —modification of Newton's law
—tidal effect
—12-hour period of tide —tide as a quadrupolar response
—solar tide and lunar tide
Constructive superposition —spring tide; destructive —neap tide.
Tidal locking.

§Inertial frames

A frame in which Newton's 1st and 2nd laws are valid is called an inertial frame. Newton's 1st law essentially states the existence of inertial frames. In practice, an ideal perfectly inertial frame does not exist. The Earth is a pretty good concrete approximation, but it spins and orbits the Sun; the Sun also moves in the Milky Way.

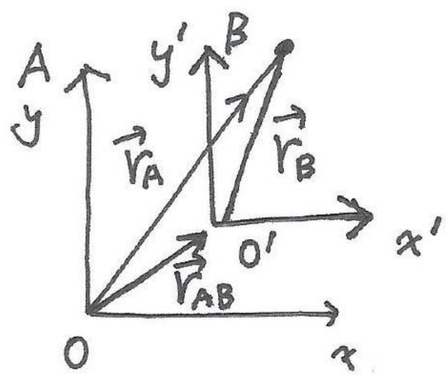
(1) Suppose reference frame A is inertial. Another frame B is static with respect to A but shifted spatially. The transformation between displacements is

$$\begin{cases} \vec{r}_A(t) = \vec{r}_{AB} + \vec{r}_B(t'), \\ t = t'. \end{cases}$$

In this case

$$\frac{d}{dt}\vec{r}_A(t) = \frac{d}{dt'}\vec{r}_B(t'), \qquad \frac{d^2}{dt^2}\vec{r}_A(t) = \frac{d^2}{dt'^2}\vec{r}_B(t'), \qquad \vec{F}_A = \vec{F}_B.$$

If in the A-frame $\vec{F}_A = m\vec{a}_A$, then $\vec{F}_B = m\vec{a}_B$ in the B-frame as well. Therefore B is also **inertial**.



(2) Suppose the frame B is rotated by angle θ with respect to A. The coordinate transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \qquad t = t'.$$

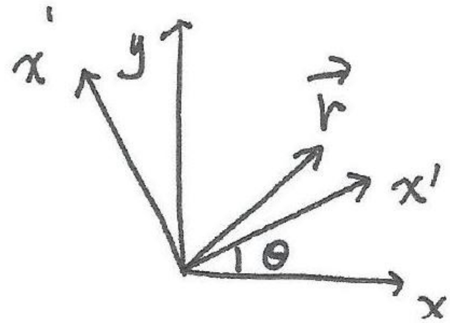
Hence

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = R_z(\theta) \begin{pmatrix} v_{x'} \\ v_{y'} \end{pmatrix}, \qquad \begin{pmatrix} a_x \\ a_y \end{pmatrix} = R_z(\theta) \begin{pmatrix} a_{x'} \\ a_{y'} \end{pmatrix},$$

and force, being a vector, transforms identically:

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = R_z(\theta) \begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix}, \qquad R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Thus if $\binom{F_x}{F_y} = m\binom{a_x}{a_y}$ holds in A, then $\binom{F_{x'}}{F_{y'}} = m\binom{a_{x'}}{a_{y'}}$ holds in B. So B is inertial.



(3) If B moves with constant velocity $\vec{v}_0$ relative to A, they are related by the Galilean boost:

$$\begin{cases} x = x' + v_0 t \\ y = y' \\ t = t' \end{cases} \Rightarrow \begin{cases} v_x = v'_x + v_0 \\ v_y = v'_y \end{cases} \Rightarrow \begin{cases} a_x = a'_x \\ a_y = a'_y \end{cases}$$

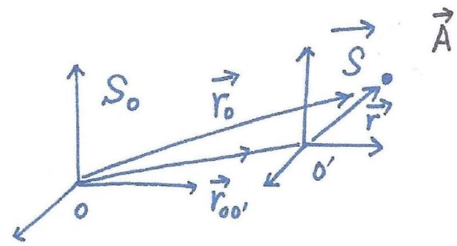
Therefore, if $\vec{F} = m\vec{a}$ is valid in A, it remains valid in B under Galilean transformations.
So far we have shown that **translations, rotations, and Galilean boosts** preserve Newton's second law. In fact, the collection of these operations forms a symmetry group -- **Galilean group**.

Galilean Relativity: Mechanical laws are the same in all inertial reference frames. No inertial frame is special. You may compare it with the **Lorentz boost** from **special relativity** in the future.

§Non-inertial frames

If a reference frame is rotating or accelerating with respect to an inertial frame, then Newton's laws are no longer valid in their simple form. It's then called a **non-inertial frame**. An accelerating car, train, or elevator is non-inertial; strictly speaking, Earth is also non-inertial due to its spin and orbital motion. It is convenient to use non-inertial frames because measurements are performed in them; then we must modify Newton's second law by adding **fictitious (inertial) forces**.

Acceleration without rotation



S_0: inertial frame; S: non-inertial frame.
$\vec{r}_0$: coordinate in S_0; $\vec{r}$: coordinate in S;
$\vec{r}_{00'}$: relative vector between S_0 and S

Newton's law in S_0:

$$m\ddot{\vec{r}}_0 = \vec{F}$$

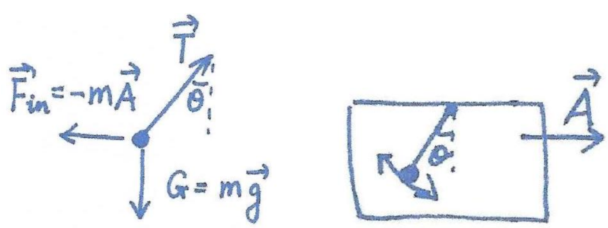
Relation between frames:

$$\vec{r}_0 = \vec{r}_{00'} + \vec{r} \Rightarrow \ddot{\vec{r}}_0 = \ddot{\vec{r}}_{00'} + \ddot{\vec{r}} = \vec{A} + \ddot{\vec{r}}.$$

in frame S:

$$m\ddot{\vec{r}} = \vec{F} - m\vec{A} = \vec{F} + \vec{F}_{\text{inertial}}, \qquad \vec{F}_{\text{inertial}} = -m\vec{A}.$$

Example: pendulum in an accelerating car



$$m\ddot{\vec{r}} = \vec{T} + \vec{G} + \vec{F}_{\text{in}} = \vec{T} + m(\vec{g} - \vec{A}), \qquad \vec{g}_{\text{eff}} = \vec{g} - \vec{A},$$

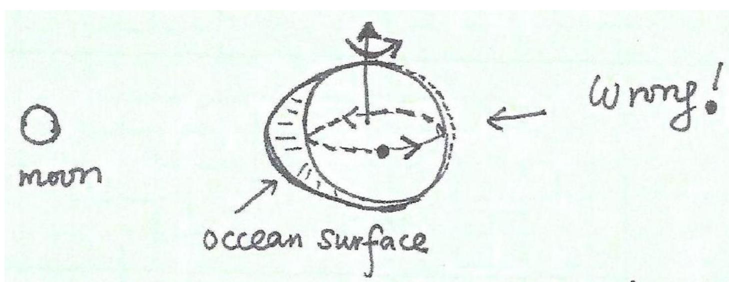
$$|\vec{g}_{\text{eff}}| = \sqrt{g^2 + A^2}, \qquad \tan\theta = \frac{A}{g}, \qquad \omega = \sqrt{\frac{g_{\text{eff}}}{l}} = \omega_0\sqrt{1 + \left(\frac{A}{g}\right)^2}.$$

Application: tides due to the Moon gravity

A famous Tang dynasty poem by Zhang Ruoxu (张若虚), *Spring River, Flower, Moon, Night* (《春江花月夜》), beautifully captures the connection between tides and the rising moon:

原文	**English Translation**
春江潮水连海平，海上明月共潮生。	The spring river rises with the sea tide, and over the sea, the bright moon ascends together with the tide.
滟滟随波千万里，何处春江无月明?	Its shimmering light follows the waves for thousands of miles— where on the river does the moon not shine bright?

This verse illustrates the poetic perception of the *moon–tide connection*, long before the physics of tidal forces was fully understood.



If the Earth were an inertial frame, the ocean surface height would show a dipolar response to the Moon's gravity (a single high tide per day) —but this is not true because the Earth is non-inertial.

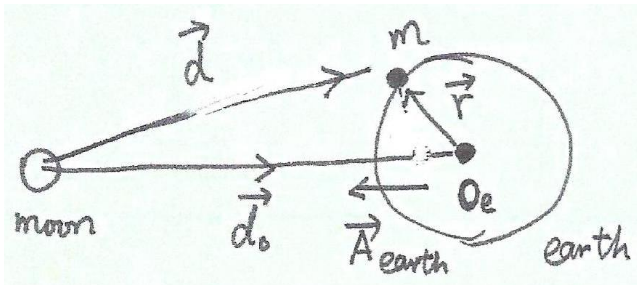
The Moon–Earth system orbits its center of mass. The Earth's acceleration is

$$\vec{A}_{\text{earth}} = -\frac{G\, m_{\text{moon}}}{d_0^2}\hat{d}_0.$$

$$\begin{cases} m\ddot{\vec{r}} = \vec{F} - m\vec{A}_{\text{earth}}, \\ \vec{F} = m\vec{g} + \vec{F}_{\text{other}} - \dfrac{GM_{\text{moon}}m}{d^2}\,\hat{d}. \end{cases}$$

Thus

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{\text{other}} - GM_{\text{moon}}m\left[\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right], \quad \vec{F}_{\text{tide}} = -GM_{\text{moon}}m\left[\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right].$$



In the Earth frame, the mass point at $\vec{r}$ feels the tidal gravity (schematic). The Earth is falling to the Moon at $\vec{A}_{\text{earth}}$.

Let $\vec{d} = \vec{d}_0 + \vec{r}$, with $|\vec{r}|/d_0 \sim 6.4 \times 10^3$ km$/3.8 \times 10^5$ km $\approx 0.02 \ll 1$. Then

$$\frac{\hat{d}}{d^2} = \frac{\vec{d}}{d^3} = \frac{\vec{d}_0 + \vec{r}}{d_0^3\left(1 + \left(\frac{r}{d_0}\right)^2 + \left(\frac{2\vec{r}\cdot\vec{d}_0}{d_0^2}\right)\right)^{3/2}} \simeq \frac{\vec{d}_0 + \vec{r}}{d_0^3\left(1 + \frac{3\vec{d}_0\cdot\vec{r}}{d_0^2}\right)} \simeq \frac{\vec{d}_0}{d_0^3} + \left[\frac{\vec{r}}{d_0^3} - \frac{3\vec{d}_0}{d_0^3}\frac{\vec{d}_0\cdot\vec{r}}{d_0^2}\right],$$

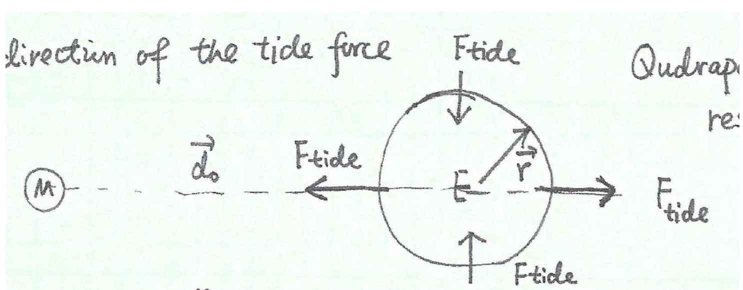
so

$$\vec{F}_{\text{tide}} = \frac{GM_{\text{moon}}m}{d_0^2}\left[-\frac{\vec{r}}{d_0} + \frac{3\hat{d}_0(\hat{d}_0\cdot\vec{r})}{d_0}\right].$$

Interpretation: the tidal force is the *gradient* (spatial variation) of the Moon's gravity —the difference between gravity at $\vec{r}$ and at Earth's center O_e. In a uniform gravity field, no tidal force:

$$\vec{F} = m\vec{g} - m\vec{A} = m(\vec{g} - \vec{g}) = 0.$$

Direction of the tidal force (12-hour response):



$$\vec{F}_{\text{tide}} = \frac{GM_{\text{moon}}\,m}{d_0^2}\underbrace{\left[-\hat{r} + 3\hat{d}_0\left(\hat{d}_0\cdot\hat{r}\right)\right]}_{\text{angular form factor}}\frac{|r|}{d_0}$$

For points on Earth:

$$\hat{r} \parallel \hat{d}_0 \Rightarrow \vec{F}_{\text{tide}} = \frac{GM_{\text{moon}}m}{d_0^3}|r|\,(2\hat{r}), \qquad \hat{r} \perp \hat{d}_0 \Rightarrow \vec{F}_{\text{tide}} = \frac{GM_{\text{moon}}m}{d_0^3}|r|\,(-\hat{r}).$$

Sun's contribution to tidal effect

The Sun also produces tides. Although $M_\odot \gg M_{\text{moon}}$, the tidal effect scales as $1/r^3$.

Solar mass $M_\odot = 2 \times 10^{30}$ kg; lunar mass $M_{\text{moon}} = 7.3 \times 10^{22}$ kg.

Earth–Sun distance $d_{\text{ES}} = 1.5 \times 10^8$ km; Moon–Earth distance $d_{\text{EM}} = 3.8 \times 10^5$ km.

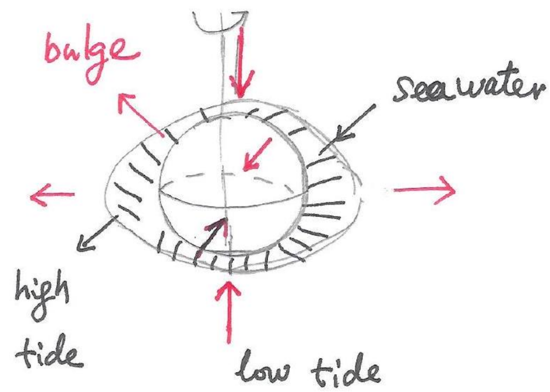
$M_\odot / M_{\text{moon}} \approx 2.7 \times 10^7$, $\quad d_{\text{ES}}/d_{\text{EM}} \approx 3.9 \times 10^2 \Rightarrow (d_{\text{ES}}/d_{\text{EM}})^3 \approx 6 \times 10^7$. Hence,

$$\frac{M_\odot / d_{\text{ES}}^3}{M_{\text{moon}} / d_{\text{EM}}^3} \simeq \frac{2.7}{6} \approx 45\%.$$

So the solar tide is about half the lunar tide.

Periodicity of tides

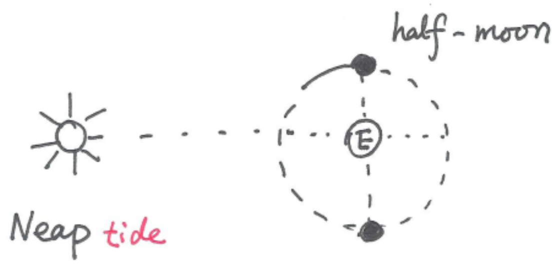
Daily tidal period is 12 hours (not 24 hours) -- quadrapolar response.



In homework, you will estimate the altitude differences between high and low tides: $\Delta h_{\text{moon}} \approx 54$ cm, and the Sun's effect is about half, $\Delta h_{\text{sun}} \approx 25$ cm. They can superpose constructively (spring tide) or destructively (neap tide).

Solar tide vs. lunar tide

$$\Delta h = \Delta h_{\text{moon}} + \Delta h_{\text{sun}} = 79 \text{ cm}$$



$$\Delta h = \Delta h_{\text{moon}} - \Delta h_{\text{sun}} = 29 \text{ cm}.$$

Tidal Locking[1]

Tidal forces not only produce periodic ocean tides, but also gradually modify the long-term dynamics of the Earth--Moon system. The key effect is that the tidal bulges raised on Earth are dragged slightly ahead of the Earth--Moon line because Earth rotates faster than the Moon orbits. This misalignment creates a tangential gravitational component, which transfers angular momentum from the Earth's spin to the Moon's orbital motion.

Energy minimization argument

Let the angular velocities be:

ω_e : Earth's spin, ω_m : Moon's spin, ω_0 : Moon's orbital angular velocity.

Denote I_e, I_m as the rotational inertias of Earth and Moon, and $I_0 = mR^2$ as the orbital inertia of the Moon (with m the Moon's mass and R the Earth--Moon distance).

The total angular momentum of the system is

$$J = I_e\omega_e + I_m\omega_m + I_0\omega_0,$$

while the total kinetic energy is

$$E_K = \tfrac{1}{2}\left(I_e\omega_e^2 + I_m\omega_m^2 + I_0\omega_0^2\right).$$

By the Cauchy inequality,

$$J^2 \leq \left(I_e + I_m + I_0\right)\left(I_e\omega_e^2 + I_m\omega_m^2 + I_0\omega_0^2\right),$$

so that

$$E_K \geq \frac{J^2}{2(I_e + I_m + I_0)}.$$

[1]Based on article by Congjun Wu, https://wuli.iphy.ac.cn/article/doi/10.7693/wl20241107

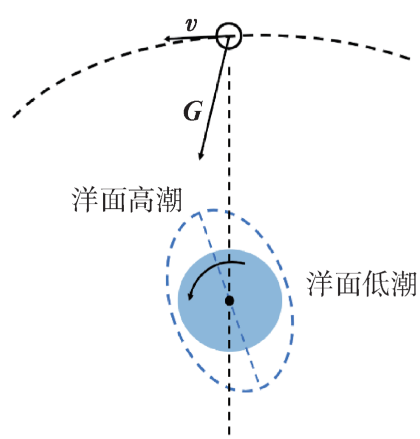
Equality holds only when

$$\omega_e = \omega_m = \omega_0,$$

i.e. when Earth's spin, Moon's spin, and the Moon's orbital motion are synchronized. This is the **tidal locking configuration**, corresponding to the minimum kinetic energy at fixed total angular momentum.

Current status of the Earth--Moon system

- The Moon is already tidally locked: $\omega_m = \omega_0$, so it always presents the same face to Earth.
- Earth is not yet locked. Its day (24 h) is much shorter than the Moon's orbital period (29.5 days), so $\omega_e \gg \omega_0$.
- Tidal friction dissipates mechanical energy into heat, while conserving total angular momentum. The Earth's rotation slows, and the Moon's orbital angular momentum increases, so the Moon slowly recedes ($\dot{R} \approx +3.8\,\mathrm{cm/yr}$).
- Fossil evidence shows that 370 Myr ago, Earth's day was only about 22 h, consistent with this gradual spin-down.



Future evolution

Over billions of years, Earth's rotation will continue to slow until $\omega_e = \omega_0$. At that point both Earth and Moon will be mutually tidally locked: one hemisphere of Earth will permanently face the Moon, and the Moon will orbit at a fixed distance. The Earth--Moon system will then behave like a rigidly rotating body with minimal internal energy at fixed J.