General Physics I

Lect6. Planetary Motion and Gravity

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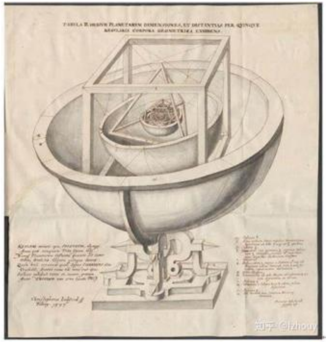
Kepler and Tycho

Newton: " I stood on the shoulders of giants." Giants = Galileo and Kepler.

Tycho Brahe collected extensive planetary data over many years. Kepler Studied his data after his death.
Discovered three fundamental laws governing planetary motions.

Johannes Kepler
(1571-1630AD)



Early Theories:
Inspiration from Platonic Solids -- five convex regular polyhedral.
- Mercury ↔ Octahedron
- Venus ↔ Icosahedron
- Mars ↔ Dodecahedron
- Jupiter ↔ Tetrahedron
- Saturn ↔ Cube

Eccentricities of Planets
- Mercury: 0.2 (most eccentric)
- Mars: 0.09
- Jupiter: 0.05
- Earth: 0.02
- Venus: 0.007

not circle ☹

Tycho Brahe
(1546-1601AD)

Overview: Kepler's laws of planetary motion

First Law
The planet's orbit is a planar ellipse, and the Sun lies at one of the ellipse's foci.

Second Law
The areas swept by the line connecting the Sun and a planet are equal in equal time intervals.

Third Law
For different orbits, the ratio between the cube of half major axis and the period square is a constant.

Impact on Astronomy:
- Provided a mathematical description of planetary orbits.
- Shifted the understanding from philosophical to empirical.

What Makes Planets Go Around?

Ancient Theories:

- Belief in invisible angels propelling planets along their paths.

Galileo's Discovery: The Principle of Inertia

- An object in motion remains in motion at a constant speed in a straight line unless acted upon by a force.
- **Key Insight:** No force is needed to keep a planet moving forward; it will continue on its own due to inertia.

Implication for Planetary Motion:

- The need for a force is not to keep planets moving forward but to change their direction.
- The force acting on planets might be directed towards the Sun, altering their straight-line paths into orbits.

Kepler's First Law

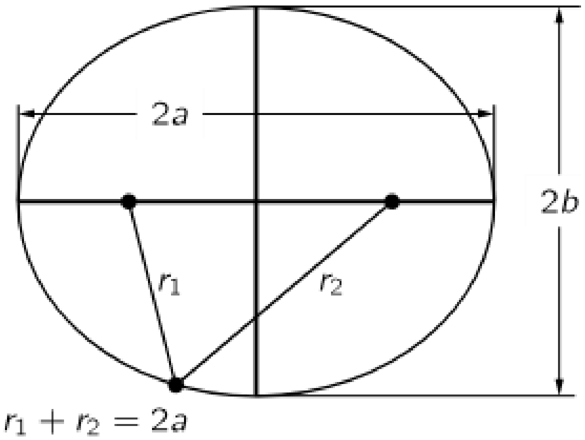
The planet's orbit is a planar **ellipse**, and the Sun lies at one of the ellipse's **foci**.

Planar Motion

- Planetary orbits are planar, lying in a two-dimensional plane.

Closed and Periodic Orbits

- Orbits are closed paths, ensuring periodic motion.



Copernicus thought that the planet orbit should be a circle as influenced by the aesthetic philosophy of the Greeks. Nevertheless, Kepler figured out in general a planet's orbit is an ellipse.

Proof of the first law using calculus

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

$$\vec{r}(t) = r(t)\,\hat{e}_r \qquad \frac{d\hat{e}_r}{dt} = \dot{\theta}\,\hat{e}_\theta, \quad \frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\,\hat{e}_r$$

$$\vec{v} = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\vec{a} = \ddot{r}\,\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

$$\vec{a} = \ddot{r}\,\hat{e}_r + \dot{r}(\dot{\theta}\,\hat{e}_\theta) + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\,\hat{e}_r)$$

- Radial acceleration:

$$a_r = \ddot{r} - r\dot{\theta}^2$$

Substitute a_r and F_r into Newton's second law:

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{GMm}{r^2}$$

Substitute $\dot{\theta} = \frac{L}{mr^2}$: (Angular Momentum Conservation)

$$\ddot{r} - \frac{L^2}{m^2r^3} = -\frac{GM}{r^2}$$

To simplify the equation, we use the substitution:

$$u = \frac{1}{r}$$

$$\dot{\theta} = \frac{L}{mr^2} = \frac{Lu^2}{m} \qquad \frac{d}{dt} = \frac{Lu^2}{m}\frac{d}{d\theta}$$

$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{d\theta}\dot{\theta}\right) \quad \ddot{r} = -\frac{L}{m}\frac{d^2u}{d\theta^2}\dot{\theta} = -\frac{L}{m}\frac{d^2u}{d\theta^2}\cdot\frac{Lu^2}{m} = -\left(\frac{L}{m}\right)^2 u^2\frac{d^2u}{d\theta^2}$$

Proof of the first law using calculus

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

$$\dot{\theta} = \frac{L}{mr^2} = \frac{Lu^2}{m} \qquad \frac{d}{dt} = \frac{Lu^2}{m}\frac{d}{d\theta}$$

$$\ddot{r} = -\frac{L}{m}\frac{d^2u}{d\theta^2}\dot{\theta} = -\frac{L}{m}\frac{d^2u}{d\theta^2}\cdot\frac{Lu^2}{m} = -\left(\frac{L}{m}\right)^2 u^2\frac{d^2u}{d\theta^2}$$

Substitute $\ddot{r}$ and $r = \frac{1}{u}$:

$$-\left(\frac{L}{m}\right)^2 u^2\frac{d^2u}{d\theta^2} - \frac{L^2u^3}{m^2} = -GMu^2$$

The equation simplifies to:

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2}$$

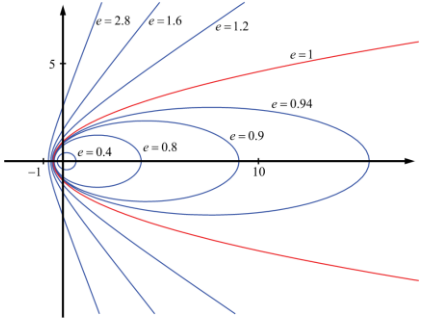
So the general solution is:

$$u(\theta) = A\cos(\theta) + B\sin(\theta) + \frac{GMm^2}{L^2}$$

Ellipse: $0 \le e < 1$

$$u(\theta) = \frac{GMm^2}{L^2}(1 + e\cos(\theta - \theta_0))$$

$$r(\theta) = \frac{p}{1 + e\cos(\theta - \theta_0)}$$



Symbol	**English name**	**中文名**
$r(\theta)$	radial distance from focus	径向距离
θ	polar angle	极角
θ_0	periapsis orientation angle	近心点方向角
e	eccentricity	偏心率
p	semi-latus rectum ($p = h^2/\mu$)	半通径

Eccentricity e	**Orbit type (English)**	**中文名**	**Key properties**
$e = 0$	Circle	圆	$r(\theta) = p = a$, radius constant
$0 < e < 1$	Ellipse	椭圆	Semi-major axis a, semi-minor axis $b = a\sqrt{1-e^2}$ periapsis $r_p = a(1-e)$, apoapsis $r_a = a(1+e)$
$e = 1$	Parabola	抛物线	Escape trajectory; $r = \frac{p}{1+\cos f}$, no apoapsis
$e > 1$	Hyperbola	双曲线	Open orbit; periapsis $r_p = \frac{p}{1+e}$, asymptotic angle defined

Kepler‘s Second Law

The areas swept by the line connecting the Sun and a planet are equal in equal time intervals.

Equal Areas in Equal Times

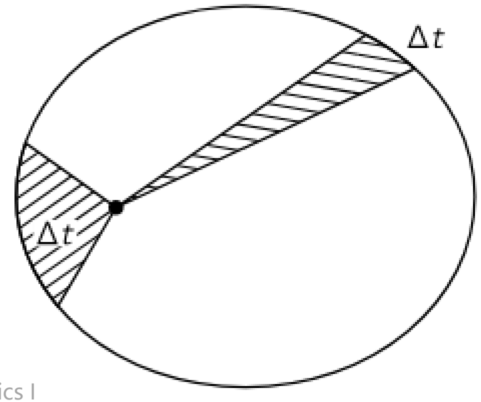
- The area swept by the planet-Sun line is constant over equal time intervals.

Variable Planetary Speed

- Planets move faster when closer to the Sun and slower when farther away.

Consequence of Angular Momentum Conservation

- The law is a manifestation of the conservation of angular momentum.



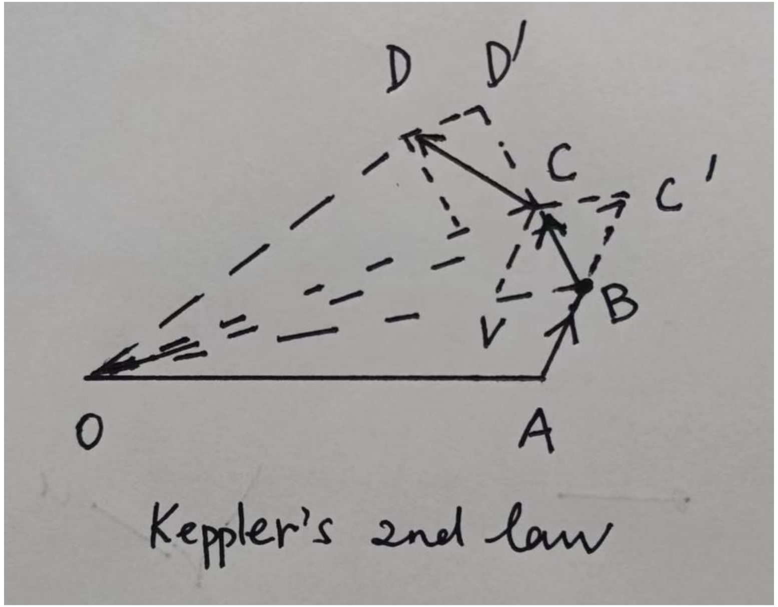
$$\Delta S = \frac{1}{2} r_1 \Delta s = \frac{1}{2} r_1 v_1 \Delta t.$$

$$\Delta S = \frac{1}{2} r v sin\theta.$$

$$m r_1 v_1 = m r_2 v_2.$$

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v}.$$

Geometric proof of the second law by Newton



In Newton's Principia, he adopted the style of **Eculid's Elements** by using the **geometric method**. At that time, the mathematical foundation of calculus was not rigorously established until the 19th century.

$$v_{BC} = \frac{BC}{\Delta t} \qquad v_{AB} = \frac{AB}{\Delta t}$$

$$S_{\Delta OAB} = S_{\Delta OBC'} \qquad AB = BC'$$

$$S_{\Delta OBC} = S_{\Delta OBC'} \qquad C'C \parallel OB$$

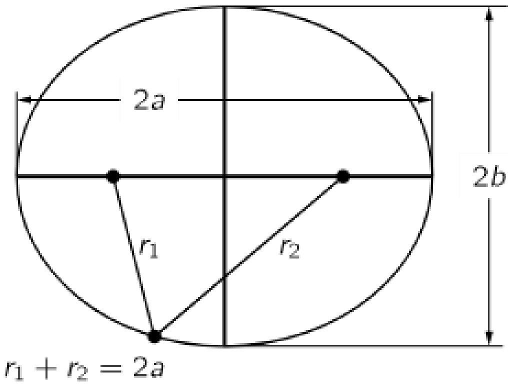
Proven $S_{\Delta OAB} = S_{\Delta OBC}$

$$S_{\Delta OAB} = S_{\Delta OBC} = S_{\Delta OCD} = S_{\Delta ODE} = \ldots$$

Kepler's Third Law

For different orbits, the ratio between the cube of half major axis and the period square is a constant.

Kepler's 3rd law implies the **inverse-square law**:
Consider the special case of a circular orbital, then a = R. Due to the nature of the periodical motion, the acceleration, roughly speaking, scales as F/m ~ v/T ~ R/T^2.
According to Kepler's 3rd law that $T^2 \sim R^3$, we arrive at $F \sim R^{-2}$



Quick "proof" of the third law

Kepler's third law can be shown by a scaling method. Suppose $\mathbf{r}(t)$ solves

$$\frac{d^2\mathbf{r}(t)}{dt^2} = -\frac{GM}{r^2}\,\mathbf{e}_r. \tag{6.22}$$

Perform the scaling

$$\mathbf{r}^s(t) = \lambda_1\,\mathbf{r}(\lambda_2 t). \tag{6.23}$$

It is easy to show

$$\frac{d^2\mathbf{r}^s(t)}{dt^2} + \frac{GM}{(r^s)^2}\,\mathbf{e}_r = \lambda_1\lambda_2^2\frac{d^2\mathbf{r}(t)}{dt^2} + \lambda_1^{-2}\frac{GM}{(r)^2}\,\mathbf{e}_r = 0, \tag{6.24}$$

provided

$$\lambda_2^2\lambda_1^3 = 1. \tag{6.25}$$

Thus the spatial size L of the orbit and the period T obey

$$L^3/T^2 = \text{const.} \tag{6.26}$$

Newton's Observation

Force Direction and Magnitude:

- Based on Kepler's Second Law, forces acting on planets are directed toward the Sun. **No angel (tangential force) is needed!**
- Kepler's Third Law suggests that such force decreases with the square of the distance.

Newton's Law of Universal Gravitation:

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e_r},$$

- F: Gravitational force
- G: Gravitational constant
- M, m: Masses of two objects
- r: Distance between centers

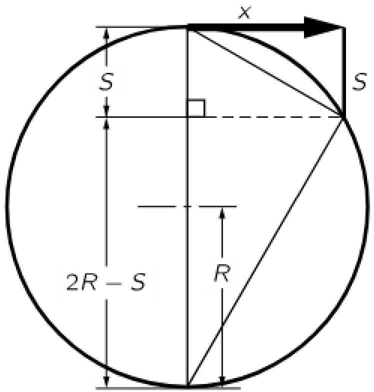
Falling Moon Analogy

Newton realized the inverse-square law by comparing the satellite motion – Moon's orbit to the free fall motion of the ground. This is probably the origin of the legend of Newton's apple.

We know that the falling distance in one second on the ground of the earth is about 5m. How much is the "falling distance" of the Moon in one second?

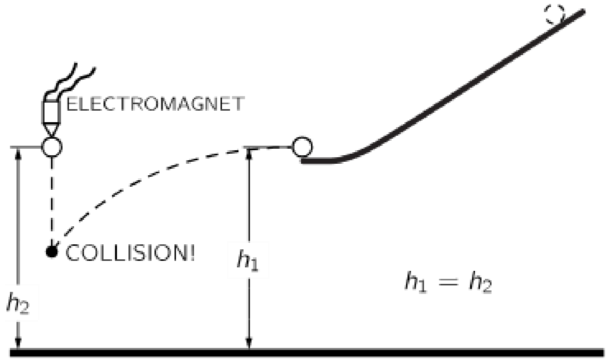
$$\frac{v\Delta t}{s} = \frac{2d_{em}}{v\Delta t}$$

$$s = \frac{1}{2}\frac{v^2}{d_{em}}t^2 = \frac{1}{2}\omega^2 d_{em}t^2 = \frac{2\pi^2}{T^2}d_{em}t^2.$$



The results of Earth-moon distance via parallax by Ptolemy, Huygens, and Tcyho Brahe are very close to each other as d_em/r_e = 59, 60, 60.5, respectively. Plugging in the period T = 27.3 days, and d_em = 60 r_e, we arrive at s =1.36mm, which is about 1/3676 =1/60^2 of the falling distance on the earth.
This is quite accurate and first unifies the forces that govern planetary motion and attraction on Earth.

Implications



Independence of Vertical and Horizontal Motions:

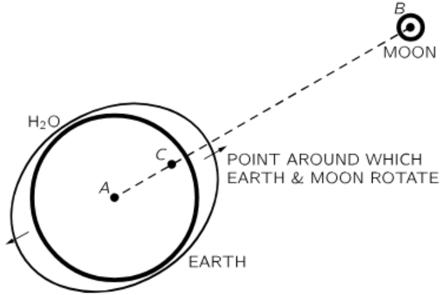
An object dropped vertically and one projected horizontally from the same height will **both fall the same vertical distance** in the same time.

At a certain speed, the projectile falls toward Earth but never gets closer, effectively "falling around" the planet. Start with 16ft/s,

$$x = \sqrt{s \times D} = \sqrt{0.00303\,\mathrm{mi} \times 8,000\,\mathrm{mi}} \approx \sqrt{24.24} \approx 4.92\,\mathrm{miles}$$

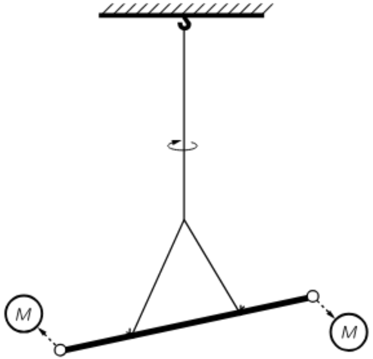
miles per second to orbit Earth at the surface level

The law of gravitation explains many phenomena not previously understood. For example, the pull of the moon on the earth causes the **tides**.



Measurements of Gravity

1797 $F = G\frac{mm'}{r^2}$



"weighing the earth"

$$6.670 \times 10^{-11}\ \text{newton} \cdot \text{m}^2/\text{kg}^2 .$$

$$\frac{\text{Gravitation Attraction}}{\text{Electrical Repulsion}} = 1/4.17 \times 10^{42}$$

Universe age / light through proton~10^{42}

2020 $V(r) = V_N(r)[1 + \alpha \exp(-r/\lambda)]$

New Test of the Gravitational $1/r^2$ Law at Separations down to 52 μm

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We tested the gravitational $1/r^2$ law using a stationary torsion-balance detector and a rotating attractor containing test bodies with both 18-fold and 120-fold azimuthal symmetries that simultaneously tests the $1/r^2$ law at two different length scales. We took data at detector-attractor separations between 52 μm and 3.0 mm. Newtonian gravity gave an excellent fit to our data, limiting with 95% confidence any gravitational-strength Yukawa interactions to ranges < 38.6 μm.