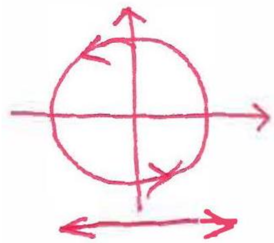
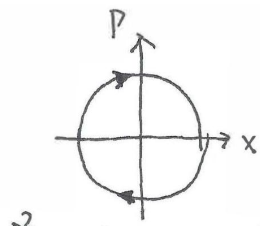
Lecture 5: Newton's laws of motion (2): Oscillation

Outline:

1. Harmonic oscillator and uniform circular motion

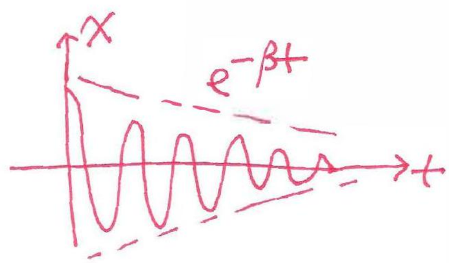


2. Phase space orbit, Bohr–Sommerfeld condition



$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$$

$$\oint p\, dx = (n + 1/2)h$$



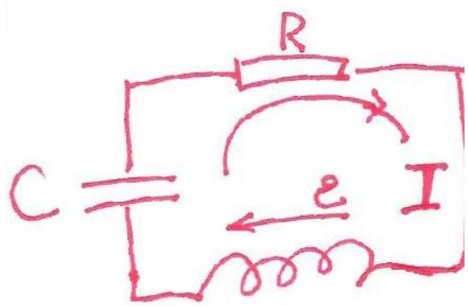
3. Damped harmonic oscillators (overdamped, underdamped, critical)

$$\ddot{x} + \frac{1}{2}\dot{x} + \omega_0^2 x = 0$$

$$x(t) = A\cos(\omega t + \phi)e^{-\beta t}$$

$$\omega = \omega_0\sqrt{1 - \left(\frac{1}{2\omega_0}\right)^2}, \quad \beta = \frac{1}{2\tau}$$

4: Cyclotron orbit:
$R = \sqrt{\frac{\hbar c}{qB}}$



5: LC oscillators

$$L \cdot \frac{d^2Q}{dt^2} + R \cdot \frac{dQ}{dt} + \frac{Q}{C} = 0.$$

Math Crash Course for Physics Learners cf. Feynman Ch.22

We start in the middle: assume integers, counting, and the idea of *add one.* From there we keep the *rules* and enlarge the *numbers* only when a problem can't be solved without doing so.

1. Addition ⇒ Multiplication ⇒ Powers. Define $a + b$, then ab as repeated addition, then a^n as repeated multiplication. The payoff is the simple algebra of exponents:

$$a^{m+n} = a^m a^n, \qquad (a^m)^n = a^{mn}.$$

2. Inverses. Solve the direct operations backwards: subtraction, division, roots, logarithms. Logs turn products into sums:

$$\log(ab) = \log a + \log b, \quad e^{x+y} = e^x e^y.$$

3. Generalize the Numbers. When an equation has no solution, enlarge the number system but keep the rules:

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}, \qquad i^2 = -1.$$

Complex numbers close algebraic equations and, miraculously, we don' t need to invent anything beyond them.

4. The Jewel. Imaginary powers define the bridge from algebra to geometry:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Hence real oscillations are just complex exponentials seen from the real axis.

5. ODEs Become Algebra. Exponentials are eigenfunctions of differentiation: if $x(t) = e^{rt}$, then $Dx = rx$ and $D^2x = r^2x$. A linear constant-coefficient ODE,

$$x'' + 2\gamma x' + \omega_0^2 x = 0,$$

reduces to the *characteristic polynomial*

$$r^2 + 2\gamma r + \omega_0^2 = 0.$$

Roots ⇒ solution form:

$$r = -\gamma \pm i\omega_d \;\Rightarrow\; x(t) = e^{-\gamma t}\big[A\cos(\omega_d t) + B\sin(\omega_d t)\big].$$

Warning: Please take a more serious math course for deeper understanding.

Harmonic oscillators

Oscillations are a general class of phenomena, such as the elastic spring oscillator, simple pendulum, and the electromagnetic analogy of LC oscillator, etc. Quantum mechanically, each quantum mode in the free case can be viewed as a harmonic oscillator, such as the photon mode, the lattice vibration—phonon modes, etc. Waves can be viewed as a series of vibration modes propagating in space–time, including mechanical wave, E&M wave, and quantum mechanical, gravitational waves.

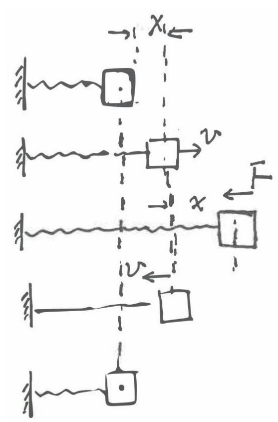
Harmonic oscillator is a prototype system to study in all branches of physics. It is simple, elegant, both in a classical way and in the quantum way. Harmonic oscillator is also the first QM problem solved by Heisenberg.

According to Hook's law, the restoring force

$$F = -kx,$$

where x is measured from its equilibrium position. Then

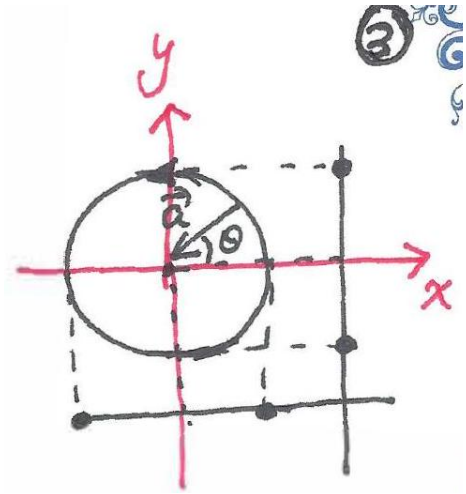
$$m\frac{d^2x}{dt^2} = -kx$$



This is a second–order constant–coefficient differential equation. Actually, we could find an elementary but smart method to solve it!

Consider a uniform circular motion, with the radius A and circular frequency ω. The acceleration

$$\begin{aligned}\vec{a} &= -a\hat{e}_r = -a(\cos\theta\hat{x} + \sin\theta\hat{y}) \\ &= -\frac{a}{A}(x\hat{x} + y\hat{y}) \\ &= -\omega^2(x\hat{x} + y\hat{y})\end{aligned}$$



let's project the motion to the x-direction, we have
$a_x = \frac{d^2x}{dt^2} = -\omega^2 x$, which is the same as the harmonic oscillator's equation with the identification $\omega = \sqrt{k/m}$.

The solution to the uniform circular motion is obvious $\theta = \omega t$. Hence

$$\begin{cases} x = A\cos(\omega t + \varphi) \\ y = A\sin(\omega t + \varphi) \end{cases}$$

where A and φ are integration constants determined by the initial conditions.

$$\begin{matrix} x(0) = A\cos\varphi \\ \dot{x}(0) = -\omega A \sin\varphi \end{matrix} \Rightarrow \begin{cases} A^2 = x^2(0) + (\dot{x}(0)/\omega)^2 \\ \tan\varphi = -\frac{\dot{x}(0)}{\omega x(0)} \end{cases}$$

Generally speaking, this kind of differential equation can be solved by trying $x(t) = e^{\lambda t}$, plugging in $\ddot{x} = -\omega^2 x \Rightarrow \lambda^2 = -\omega^2 \Rightarrow \lambda = \pm i\omega$. Hence,

$$x(t) = a_1 e^{i\omega t} + a_2 e^{-i\omega t}$$

If we require $x(t)$ is real, then $a_1 = a_2^* = Ae^{i\varphi}$, then $\Rightarrow x(t) = A\cos(\omega t + \varphi)$.

$$v(t) = \dot{x}(t) = -A\omega\sin(\omega t + \varphi)$$

We have seen the relationship between uniform circular motion and harmonic oscillation. Actually, the relation is even closer if we organize

$$-v(t)/(A\omega) = \sin(\omega t + \varphi) = y(t)$$

Hence, the y-axis motion in fact reflects *the velocity of the oscillation!*

Phase space orbits

In the canonical version of classical mechanics, momentum $\vec{p} = m\vec{v}$ is also a fundamental quantity. Based on the above reasoning,

$$\begin{cases} X = A\cos(\omega t + \varphi) \\ P = mv = -mA\omega\sin(\omega t + \varphi) \end{cases}$$
$$\Rightarrow \left(\frac{X}{A}\right)^2 + \left(\frac{P}{mA\omega}\right)^2 = 1$$

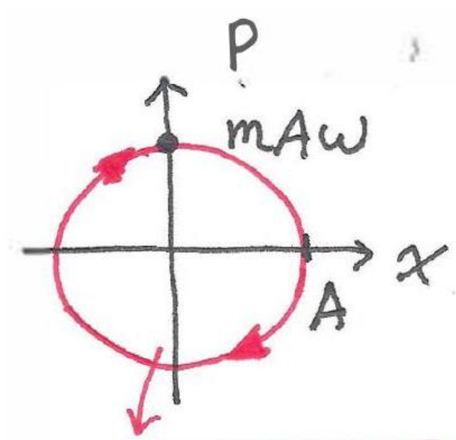


Figure 1: circular (elliptical) orbit in the phase space

Please note that the phase space orbit is chiral, i.e., it only rotates in one direction, but not in the reverse direction!
cf. *the cyclotron motion of electrons in the magnetic field.*

Classically, the area enclosed by the orbit is arbitrary. Quantum mechanically, however, the area has a minimum $\frac{h}{2}$.

Bohr-Sommerfeld quantization condition

$$\oint p\,dx = (n + 1/2)\,h, \quad n = 0, 1, 2, \ldots \tag{1}$$

over a period (2)

Minimal area :

$$\pi \cdot A_{\min}^2 \omega m = \frac{h}{2}$$
$$\Rightarrow l_0 = A_{\min} = \sqrt{\frac{\hbar}{m\omega}}$$

Hence, for a harmonic oscillation, there exists a minimum length scale l_0.

Uncertainty principle:

Since the orbit area has a minimum, it means oscillators cannot be at rest. The uncertainty $\sqrt{\langle \Delta x^2 \rangle}\sqrt{\langle (\Delta p)^2 \rangle}$ is of the same order as the area of the minimum orbit. Roughly speaking

$$\sqrt{\langle (\Delta x)^2 \rangle}\sqrt{\langle (\Delta p)^2 \rangle} \sim \hbar$$

a more precise calculation based on quantum mechanics shows

$$\sqrt{\langle (\Delta x)^2 \rangle}\sqrt{\langle (\Delta p)^2 \rangle} \geq \frac{\hbar}{2}$$

Conservation law:

We have seen $\left(\frac{p}{mA\omega}\right)^2 + \left(\frac{x}{A}\right)^2 = 1 \Rightarrow \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$.
The first term is the kinetic energy: $\frac{p^2}{2m} = \frac{1}{2}mv^2$.
The second term is determined by the status (location x) of the spring.

Let's calculate the work done to change the spring length to x:

$$\begin{aligned} & W = \int_0^x F\, dx' = k\int_0^x x' dx' = \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2 x^2 \\ \Rightarrow \quad & \text{kinetic energy + potential energy = total energy.} \end{aligned}$$

Damped harmonic oscillators:

Consider a harmonic oscillator, but with linear resistance.

$$F = m\ddot{x} = -m\omega_0^2 x - b\dot{x} \Rightarrow \ddot{x} + \omega_0^2 x + \frac{1}{\tau}\dot{x} = 0$$

where $\tau = m/b$.

(1) Trying solution:

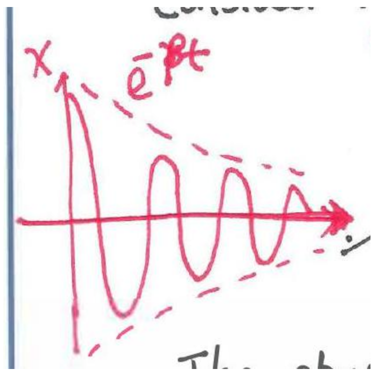
$x(t) \sim e^{i\lambda t} \Rightarrow -\lambda^2 + \frac{i\lambda}{\tau} + \omega_0^2 = 0$

$$\lambda_{1,2} = \pm\sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2} + \frac{i}{2\tau}$$

Hence,

$$x(t) = \left(a_1 e^{i\omega t} + a_2 e^{-i\omega t}\right) e^{-\frac{t}{2\tau}}, \text{ where } \omega = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2}, \beta = 1/2\tau$$

$\Rightarrow$*friction decreases frequency*



Consider the real solution:

$$x(t) = A\cos(\omega t + \phi)e^{-\beta t}, \text{ with } \omega = \omega_0\sqrt{1 - \left(\frac{1}{2\omega_0\tau}\right)^2}$$

The above solution works at $\frac{1}{2\tau} < \omega_0$, i.e., $\omega_0\tau > 1/2$. This situation is called the **underdamped** case.

Suppose we know the initial condition x_0 and v_0 at t=0, how to determine a_1, a_2?

$$x_0 = a_1 + a_1^* \leftarrow \text{ choose } a_2 = a_1^*, a_1 = a_R + ia_I$$
$$\frac{dx}{dt} = \left(ia_1\omega e^{i\omega t} - ia_1^*\omega e^{-i\omega t}\right)e^{-t/2\tau} + \left(\frac{-1}{2\tau}\right)x(t)$$
$$v_0 = i\omega\left(a_1 - a_1^*\right) - \frac{1}{2\tau}\left(a_1 + a_1^*\right)$$
$$\Rightarrow \begin{array}{ll} x_0 = 2a_R & \Rightarrow a_R = \frac{x_0}{2} \\ v_0 = -2a_I\omega - \frac{1}{\tau}a_R & \Rightarrow a_I = \frac{v_0 + \frac{x_0}{2\tau}}{2\omega} \end{array}$$

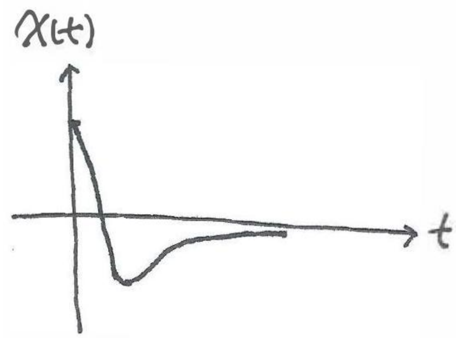
$$\Rightarrow x = e^{-t/2\tau}\left[x_0\cos\omega t + \frac{v_0 + x_0/2\tau}{\omega}\sin\omega t\right]$$

Quality factor $Q = \omega\tau \approx \omega_0\tau$ (how many turns of oscillations before half-decay)

(2) On the other hand, if $\omega_0\tau < 1/2$ (i.e., $\sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2} < 0$), there will be *transient solutions.*
Try $x(t) \sim e^{-\lambda t}$, $\lambda^2 - \lambda/\tau + \omega_0^2 = 0$

$$\lambda_{1,2} = \frac{1}{2\tau} \pm \sqrt{\left(\frac{1}{2\tau}\right)^2 - \omega_0^2}$$
$$X(t) = a_1e^{-\lambda_1 t} + a_2e^{-\lambda_2 t}$$

Plug in the initial conditions x_0 and $v_0 \Rightarrow \begin{cases} a_1 = \frac{\lambda_1 x_0 + v_0}{\lambda_1 - \lambda_2} \\ a_2 = -\frac{\lambda_2 x_0 + v_0}{\lambda_1 - \lambda_2} \end{cases}$



This situation is called the **overdamped** case.

(3) **Critical case** $\omega_0 \tau = 1/2, \quad \lambda = \frac{1}{2\tau}$

$$x(t) = a_1 e^{-\lambda t} + a_2 t e^{-\lambda t}$$

$$\begin{cases} x_0 = a_1 \\ v_0 = -\lambda a_1 + a_2 \Rightarrow a_2 = v_0 + \frac{x_0}{2\tau} \end{cases}$$

$$\Rightarrow x(t) = x_0 e^{-\frac{t}{2\tau}} + \left(v_0 + \frac{x_0}{2\tau}\right) t e^{-\frac{t}{2\tau}}$$

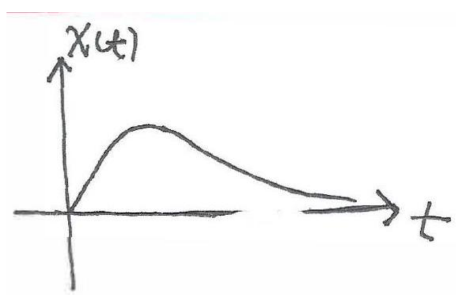
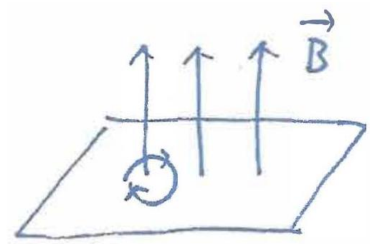


Figure 2: Critical damping decays quickest. Here we show a sketch when $x_0 = 0$.

Other examples of oscillations

I. Motion in a uniform magnetic field

$$\vec{F} = \frac{q}{c}\vec{v} \times \vec{B}. \quad \text{(Gaussian unit)} \quad \text{set } \vec{B} = B\hat{z}$$



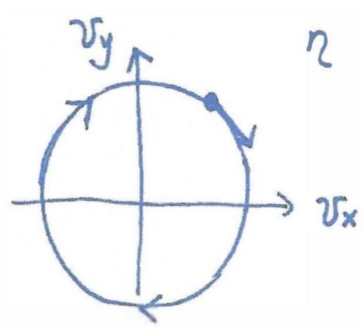
$$\begin{cases} m\dot{v}_x = \frac{q}{c}Bv_y \\ m\dot{v}_y = -\frac{qB}{c}v_x \\ m\dot{v}_z = 0 \end{cases}$$

$\rightarrow \omega = \frac{gB}{mc}$ cyclotron frequency

$$\Rightarrow \begin{cases} \dot{v}_x = \omega v_y \\ \dot{v}_y = -\omega v_x \end{cases} \Rightarrow \dot{v}_x + i\dot{v}_y = -i\omega(v_x + iv_y)$$

Define $\eta = v_x + iv_y \Rightarrow \dot{\eta} = -i\omega\eta$

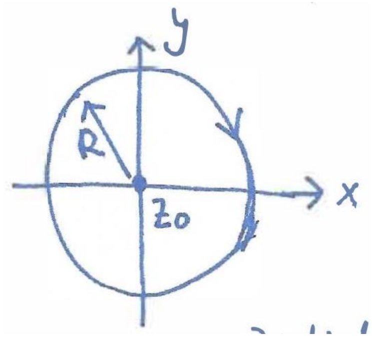
$$\Rightarrow \eta = Ae^{-i\omega t} \text{ with } A = v_x(0) + iv_y(0)$$



We also define $z = x + iy$.

$$\Rightarrow z = \text{const.} + \int^t \eta \, dt$$

$$= z_0(\text{center of circle}) + \frac{iA}{\omega}e^{-i\omega t}$$



The cyclotron radius R

$$R = \left|\frac{A}{\omega}\right| = \frac{vmc}{qB}.$$

$\Rightarrow$ momentum magnitude $P = \frac{qBR}{c}$.

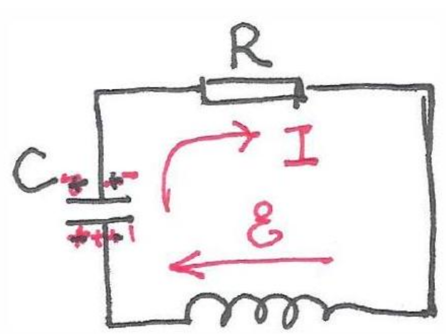
$$\rightarrow \text{ orbital angular momentum } L = PR = \frac{qBR^2}{c}$$

Classically, the circular orbit can be of any size.
But quantum mechanically, it has a minimal size.
The orbital angular momentum

$$L_{\min} = \frac{qBR^2}{c} = \hbar \Rightarrow R = \sqrt{\frac{\hbar c}{qB}} \leftarrow \text{the cyclotron radius.}$$

II. LC oscillators

$$\varepsilon = -L\frac{dI}{dt} = IR + \frac{Q}{C}$$
$$\Rightarrow L\frac{dI}{dt} + IR + \frac{Q}{C} = 0$$
$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$$



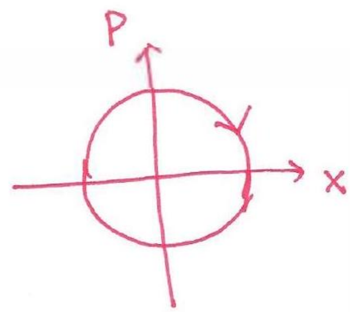
The analogy to momentum P follows the mapping:

$$\begin{array}{ccc} P = mv & \to & LI = \Phi \\ x & \to & Q \\ \dot{x} & \to & I \end{array}$$

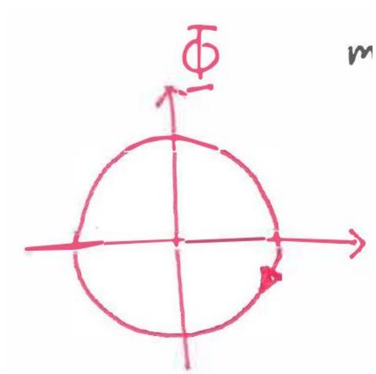
We arrived at the harmonic oscillator in an electric circuit

$$\begin{cases} L \to m,\ \frac{1}{C} \to k \\ \frac{R}{L} = \frac{1}{\tau} \Rightarrow \omega_0^2 = k/m = \frac{1}{LC} \end{cases}$$

$$\left(\frac{x}{A}\right)^2 + \left(\frac{P}{m\omega A}\right)^2 = 1$$



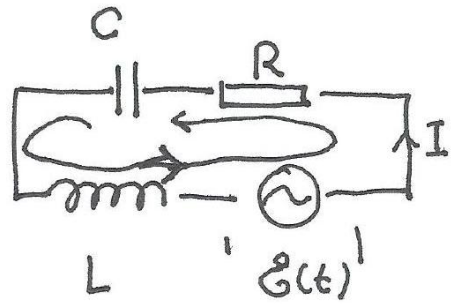
$$\left(\frac{Q}{Q_0}\right)^2 + \left(\frac{\Phi}{LQ_0\omega_0}\right)^2 = 1$$



magnetic energy ⇌ electric energy

§ Driven damped oscillations

Consider the LC circuit: in addition to the inductor, capacitor, and resistor, there is also a driving EMF $\varepsilon(t)$.



The governing equations are

$$IR + \frac{Q}{C} = L\,\frac{dI}{dt} + \varepsilon(t),$$
$$L\,\frac{d^2Q}{dt^2} + R\,\frac{dQ}{dt} + \frac{Q}{C} = \varepsilon(t),$$
$$\frac{1}{\tau} \equiv \frac{R}{L}, \qquad \omega_0^2 \equiv \frac{1}{LC}.$$

For a mechanical oscillator (mass m and external force F),

$$\ddot{x} + \frac{1}{\tau}\,\dot{x} + \omega_0^2\,x = f(t), \qquad f(t) \equiv \frac{F(t)}{m}.$$

These are the same constant-coefficient, linear, inhomogeneous ODEs. By superposition,

$$x(t) = x_h(t) + x_p(t)$$

where $x_h(t)$ solves the homogeneous equation

$$\ddot{x}_h + \frac{1}{\tau}\,\dot{x}_h + \omega_0^2\,x_h = 0$$

and $x_p(t)$ is any particular solution of

$$\ddot{x}_p + \frac{1}{\tau}\,\dot{x}_p + \omega_0^2\,x_p = f(t) \quad \longleftarrow \text{ steady solution.}$$

Because the homogeneous part decays in time, the long-time behavior is determined by $x_p(t)$.

Sinusoidal drive:

Take $f(t) = f_0 \cos\omega t$. Promote to the complex field by also considering

$$\ddot{y} + \frac{1}{\tau}\,\dot{y} + \omega_0^2\,y = f_0 \sin\omega t, \quad \text{and define } z = x + iy.$$

Then z obeys

$$\ddot{z} + \frac{1}{\tau}\,\dot{z} + \omega_0^2\,z = f_0 e^{i\omega t}.$$

Try a particular solution $z_p(t) = C\,e^{i\omega t}$. Substituting gives

$$C\left[-\omega^2 + \frac{i\omega}{\tau} + \omega_0^2\right] = f_0,$$

$$\Rightarrow \quad C = \frac{f_0}{\omega_0^2 - \omega^2 + i\,\omega/\tau}.$$

Write $C = Ae^{-i\delta}$ with

$$A^2 = \frac{f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega/\tau)^2}, \qquad \delta = \tan^{-1}\left(\frac{\omega/\tau}{\omega_0^2 - \omega^2}\right).$$

Hence

$$x_p(t) = \Re\left[Ae^{i(\omega t - \delta)}\right] = A\cos(\omega t - \delta).$$

The full solution is

$$x(t) = A\cos(\omega t - \delta) + \underbrace{c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}_{\text{transient}},$$

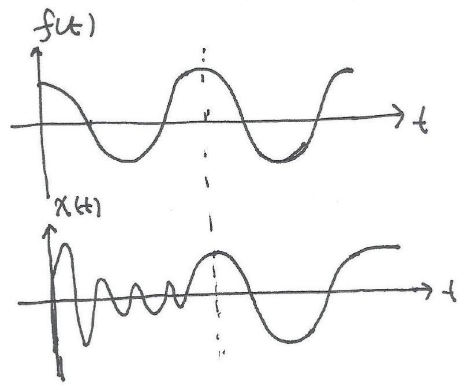
which decays to the steady motion as $t \to \infty$. For a weakly damped system ($1/\tau \ll \omega_0$),

$$x(t) = A\cos(\omega t - \delta) + e^{-t/(2\tau)}\left[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)\right],$$

where the damped natural frequency is

$$\omega_d = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2}.$$

Transient motion depends on initial conditions but decays; different initial conditions lead to the same steady motion (attractor).



§Resonance of a driven oscillator

Under the driving force $f(t) = f_0 e^{i\omega t}$ we have

$$z_p(t) = \frac{f_0\, e^{i\omega t}}{\omega_0^2 - \omega^2 + i\,\omega/\tau}.$$

Asymptotic limits.

$$\omega \to 0: \quad z_p(t) \to \frac{f_0}{\omega_0^2}\, e^{i\omega t},$$

$$\omega \to \infty: \quad z_p(t) \to -\frac{f_0}{\omega^2}\, e^{i\omega t} \qquad \text{(small amplitude).}$$

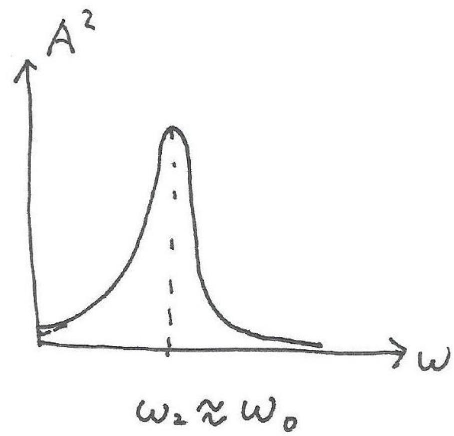
Amplitude and peak.

The complex amplitude is

$$A(\omega) = \frac{f_0}{\omega_0^2 - \omega^2 + i\,\omega/\tau}, \qquad |A(\omega)|^2 = \frac{f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega/\tau)^2}.$$

Maximal response occurs at

$$\frac{\partial}{\partial(\omega^2)}\left[\left(\omega_0^2 - \omega^2\right)^2 + (\omega/\tau)^2\right] = 0 \;\Rightarrow\; \omega_r = \sqrt{\omega_0^2 - \frac{1}{2\tau^2}} \approx \omega_0\left(1 - \frac{1}{4\omega_0^2\tau^2}\right).$$



Phase.

- For $\omega \ll \omega_0$: $\delta \to 0$ (in-phase; A is real).
- For $\omega \gg \omega_0$: $\delta \to \pi$ (out of phase by π; A is real).

Dissipation produces a frequency-dependent phase delay.

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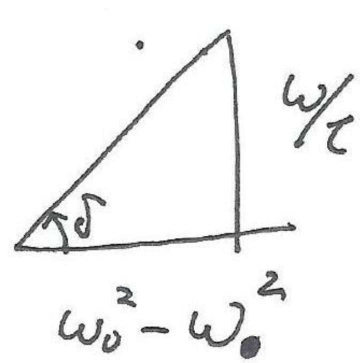


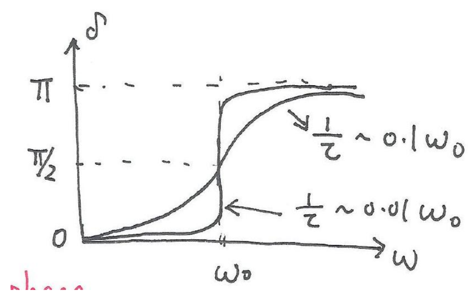
Width of the resonance (weak damping $1/\tau \ll \omega_0$).

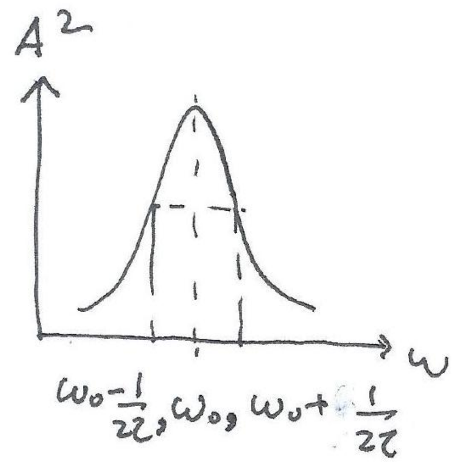
Near $\omega \approx \omega_0$, write $\omega = \omega_0 + \Delta$ so that $\omega_0^2 - \omega^2 \approx -2\omega_0\Delta$. The half-maximum condition $|A(\omega)|^2 = \frac{1}{2}|A(\omega_0)|^2$ gives

$$4\omega_0^2\Delta^2 = \left(\frac{\omega_0}{\tau}\right)^2 \;\Rightarrow\; \Delta = \pm\frac{1}{2\tau}.$$

Thus the full width at half maximum is

$$\Gamma \equiv \omega_+ - \omega_- \approx \frac{1}{\tau}, \qquad \omega_\pm \approx \omega_0 \pm \frac{1}{2\tau}.$$





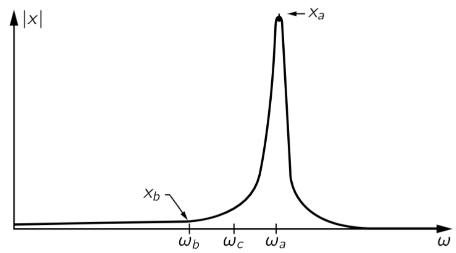


Figure 3: Feynman Fig. 25–3. A sharply tuned resonance curve.

Application of Superposition and Resonance: Radio Tuning

Radio antennas are driven by multiple oscillating EM fields, e.g. F_a at ω_a and F_b at ω_b. By superposition, the circuit receives $x_a + x_b$. Yet a resonant LC circuit responds sharply near its natural frequency ω_0. If tuned to ω_a, then $x_a \gg x_b$, so the output is dominated by station a. Changing C (or L) shifts $\omega_0 = 1/\sqrt{LC}$, allowing one to silence both or tune to ω_b. Thus, radio tuning illustrates superposition filtered by resonance.

Supplement: Linear Systems (Feynman Ch. 25)

Suppose a system is driven by F_a (e.g. oscillatory with $\omega = \omega_a$), producing response x_a. For another force F_b, the response is x_b. If both act together, the solution is simply

$$L(x_a + x_b) = L(x_a) + L(x_b) = F_a(t) + F_b(t), \tag{25.8}$$

so the total motion is $x_a + x_b$. This is the **principle of superposition**: a complicated force may be decomposed into simpler pieces, solved individually, and recombined.

An analogous result appears in electrostatics. A charge distribution q_a gives field $\mathbf{E}_a$, while q_b produces $\mathbf{E}_b$. Together,

$$\mathbf{E} = \mathbf{E}_a + \mathbf{E}_b,$$

so the total field is the vector sum. This works because Maxwell’s equations are linear.

In summary, linear systems are powerful because they allow complex problems to be solved piece by piece and then assembled into the full solution.

Application of Superposition: Methods for Complicated Forces

To treat more complex forces, two general methods are used:

- **Fourier Analysis:** A general force can be expanded as a superposition of sinusoidal components,

$$F(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{i\omega t}\, d\omega,$$

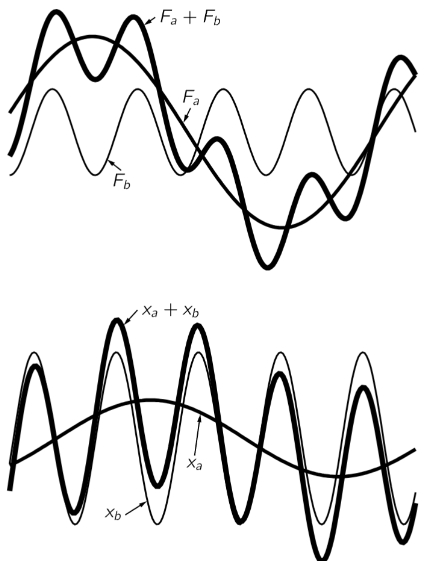


Figure 4: Feynman Fig. 25–1.An example of the principle of superposition for linear systems.

where $\tilde{F}(\omega)$ are the Fourier amplitudes. Since the response to $e^{i\omega t}$ is known, the total motion is

$$x(t) = \int_{-\infty}^{\infty} H(\omega)\,\tilde{F}(\omega) e^{i\omega t}\,d\omega,$$

with $H(\omega)$ the frequency response of the system.

- **Green' s Function:** If the response to a unit impulse $\delta(t)$ is the Green' s function $G(t)$, then for arbitrary forcing

$$F(t) \quad\longrightarrow\quad x(t) = \int_{-\infty}^{\infty} G(t-\tau)\,F(\tau)\,d\tau.$$

For example, for a damped oscillator the impulse response $G(t)$ is a decaying sinusoid.

Both approaches rely on linearity: superposition of solutions mirrors superposition of forces. Because many fundamental laws (e.g. Maxwell' s equations, Schrödinger' s equation) are linear, these methods are of central importance in physics and engineering.