Lecture 4：Newton's laws of motion(I)

Outline：

- Newton's standing on Galileo's shoulders

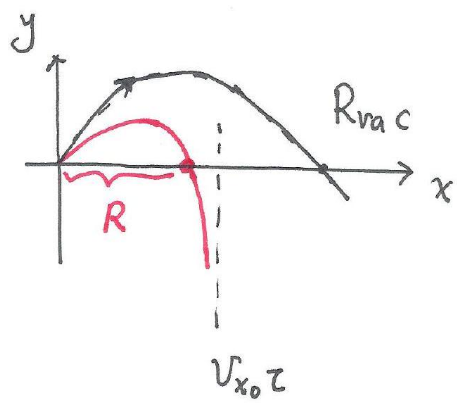
  Law of inertia

  Newton's 2nd law-concepts of force, mass

- Projectile motion with resistance

  linear vs. quadratic drags

$$\begin{cases} m\dot{v}_x = -bv_x & \tau = m/b \\ m\dot{v}_y = -mg - bv_y & v_\infty = mg/b \end{cases}$$



$R \approx R_{\text{vac}} \left[1 - \frac{4}{3}\frac{v_{y0}}{v_{(\infty)}}\right]$

Newton was remarkable!–more progress than Galileo

Galileo : discovery of the principle of inertia (Newton's first law of motion). If a body is not exerted a force on it, it keeps still, or it maintains the motion at a constant velocity.

However, this law is more like a definition. It's valid in special frames called inertial frames, but not in all the frames. For example, in the frame of an accelerated train, a free body will also accelerate. But then how to judge a frame is inertial? We will say, look at a body, if it has no force acting on it, and it's either at rest on moves with then the frame is inertial! a constant speed?This sounds like a circular reasoning! Hence, the true meaning of the "law of inertia" is the assumption of the existence of the inertial frame! It's role is like the 5 prepositions of Euclid "elements".

Galileo also realized that force results in acceleration, and acceleration is proportional to force. But he did not propose the concept of mass, and the concept of force was concrete, such as the gravity of free fall, and other daily life examples.

Standing on the shoulders of Galileo, Newton improved the concept of force to a much more general and abstract level. Using his imagination, Newton generalized the gravity for falling apples to the motions of moon, planets, etc! Newton also proposed the concept of "mass" as a measurement of inertia. Then we can compare the accelerations among different objects and different forces!

Newton's 2nd law: Time-rate-of-change of momentum is proportional to force $\vec{F} = \frac{d}{dt}(m\vec{v})$.

For non-relativistic physics, $v \ll c$, the mass can be approximated as a constant. In this case, Newton's law becomes

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2} = m\vec{a}$$

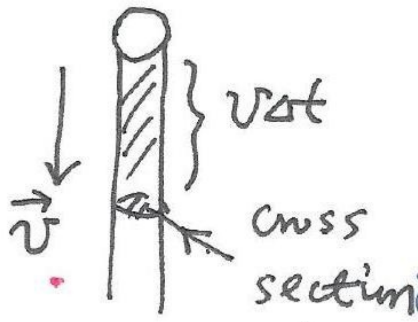
This is a 2nd order differential equation, and also a vector equation. Acceleration not only could mean the change of speed, but also could mean the change of motion direction!

Example: projectile motion with resistance

If we have the knowledge of force, in principle, we should be able to solve the equation of motion. Now we can consider a more realistic projectile motion by considering the air resistance $\vec{f} = -f(v)\hat{v}$, which is always in the opposite direction of velocity consing dissipation. (The dissipation power $\vec{f} \cdot \vec{v} < 0$).
When the speed v is small, f is proportional to v, which is called Stokes' law of viscosity.

$$f_{lin} = 3\pi\eta Dv,$$

where η is the viscocity of the fluid, D is the droplet diameter. The unit of η is $\mathrm{kg}/(\mathrm{m}\cdot\mathrm{s})$.
Actually, the resistance also contains the quadratic part. In a short time Δt, the projectile travels at a distance $v\Delta t$. It pushes the air of the volume $\mathrm{v}A\Delta t$ to the velocity v.



x is a coefficient, $0 < x < 1$.

$$f_q \cdot \Delta t = \kappa\rho_{\mathrm{air}} v A \Delta t \cdot v \Rightarrow f_q = \frac{\kappa\pi}{4} D^2 \rho_a v^2$$

Which part is more significant?

$$f_q / f_{\mathrm{lin}} = \frac{\kappa\pi \cdot D^2 \rho_a v^2}{4 \cdot 3\pi\eta D v} = \frac{\kappa}{12} \frac{D v \rho_a}{\eta} \leftarrow \text{Regnolds number}$$

plug in $\eta = 1.7 \times 10^{-5}\ \mathrm{kg}/(\mathrm{m}\cdot\mathrm{s})$, $\quad \rho_a = 1.29\ \mathrm{kg}/\mathrm{m}^3$

$$\Rightarrow f_a / f_{lin} = \left[\frac{6 \times 10^3\ \mathrm{s}}{\mathrm{m}^2}\right] \kappa D v. \quad \text{We choose a commenly used}$$
$$\rightarrow 1.5 \times 10^3 \left(\mathrm{s}/\mathrm{m}^2\right) D v$$

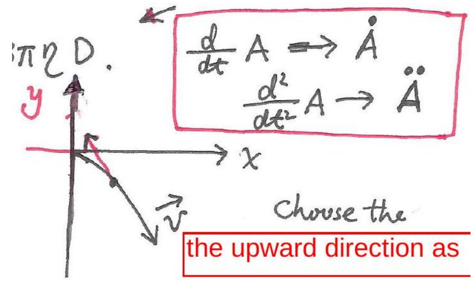
-base ball $D = 7$ cm and $v = 5$ m/s $\Rightarrow f_q / f_{\mathrm{lin}} = 600$
-rain droplet $D = 1$ mm, $v = 0.6$ m/s $\quad f_q / f_{l\ \mathrm{lin}} \simeq 1$
-Milikan oil drop $D = 1.5\mu m, v = 5 \times 10^{-5}$ m/s $\quad f_q / f_{\mathrm{lin}} \simeq 10^{-7}$.

For big and fast projectiles, the quadratic drag is more important, while for small and slow projectiles, the linear one is more important.

Motion with the linear air resistance

$$m\ddot{\vec{r}} = m\vec{g} - b\vec{v}, \quad = m\dot{\vec{v}} \Rightarrow \begin{cases} m\dot{v}_x = -bv_x \\ m\dot{v}_y = -mg - bv_y \end{cases}$$

where b=$3\pi\eta D$. Note Newton's notation here.



the upward direction as positive direction

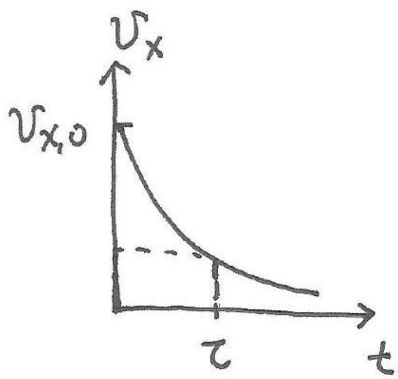
In the x-direction

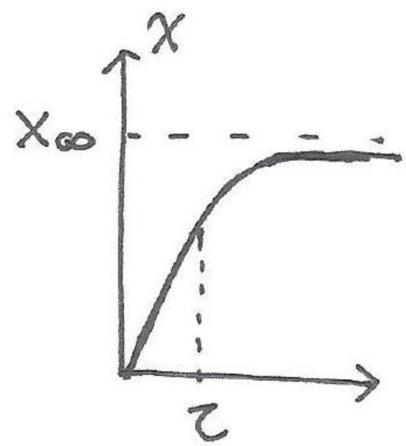
$\frac{dv_x}{v_x} = \frac{-b}{m} dt \Rightarrow v_x(t) = Ae^{-t/\tau}$,

where $\tau = m/b \leftarrow$ time constant.
Combine with $v_x(t=0) = v_0 \Rightarrow A = v_{x,0}$.

$$v_x = \frac{dx}{dt} \Rightarrow x(t) = x(0) + \int_0^t v_x dt = x(0) + v_{x,0} \int_0^t e^{-t'/2} dt'$$

Set $x(0) = 0 \Rightarrow x(t) = x_\infty \left(1 - e^{-t/\tau}\right)$, where $x_\infty = v_{x,0}\tau$.





Along the y-direction, it's an inhomogenons 1st order ODE:
$\dot{v}_y = -g - b/mv_y$.
It's solution at $t \to \infty, \dot{v}_y = 0 \Rightarrow v_y(t \to \infty) = -mg/b \triangleq -v(\infty)$

$$\Rightarrow v_y(t) = Ae^{-t/\tau} - v(\infty)$$

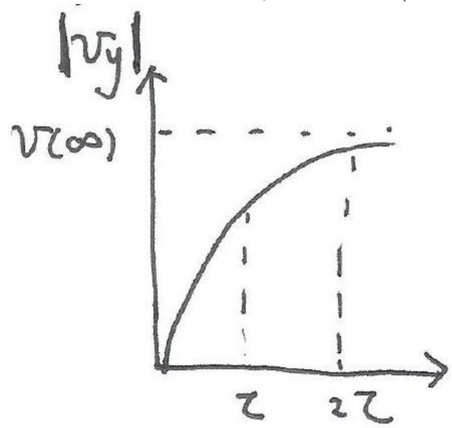
$\uparrow$
solution to the homogeneous part + a special solution
plug in $v_y(t=0) = v_{y,0} \Rightarrow A = v_{y,0} + v(\infty)$

$$\Rightarrow v_y(t) = v_{y,0}e^{-t/\tau} - v(\infty)\left(1 - e^{-t/\tau}\right)$$

$$y(t) = \int_0^t v_y(t')\,dt' = v(\infty)t + (v_{y_0} - v(\infty))\,\tau\left(1 - e^{-t/\tau}\right)$$

(set y(t=0)=0)

If $v_y(t=0) = 0$, we have $v_y = -v(\infty)\left(1 - \bar{e}^{t/\tau}\right)$



time	percent of $v(\infty)$
τ	63%
2τ	86%
3τ	95%

Estimation of orders

(1) $v(\infty) = \frac{mg}{b} = \frac{\rho\pi D^3 g}{6\cdot D3\pi\eta} = \frac{\rho\cdot D^2 g}{18\eta}$

For an oil drop in the Millikan experiment,

$D = 1.5\mu$m. $\quad \rho = 840$ kg/m$^3 \Rightarrow v(\infty) = 6.1\times10^{-5}$ m/s.

But for size $D = 0.2$ mm $\Rightarrow v(\infty) = 1.3$ m/s of drizzle drop

(2) time scale: $\tau = m/b = \frac{v(\infty)}{g}$

For millikcan drop $\Rightarrow \tau \approx 6\times10^{-6}$ s

For drizzle drop $\rightarrow \tau \approx 0.13$ s.

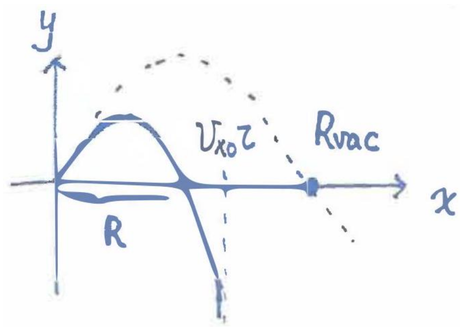
Trajectory and range

$$\left.\begin{aligned}&(1)\ x(t) = v_{x_0}\tau\left(1 - e^{-t/\tau}\right)\\ &(2)\ y(t) = (v_{y_0} + v(\infty))\,\tau\left(1 - e^{-t/\tau}\right) - v(\infty)t\end{aligned}\right\}\text{ with } x(0) = y(0) = 0.$$

From (1) $\Rightarrow t = -\tau\ln\left(1 - \frac{x}{v_{x_0}\tau}\right)$

$$y = \frac{v_{y_0} + v_\infty}{v_{x_0}}x + v(\infty)\tau\ln\left(1 - \frac{x}{v_{x_0}\tau}\right)$$

x cannot exceed $x_\infty = v_{x_0}\tau$.



The range on the horizontal direction in the vacuum

$$R_{vac} = \frac{2v_{x_0}v_{y_0}}{g}.$$

Now with resistance, we solve $\frac{v_{y_0}+v_x(\infty)}{v_{x_0}}R + v_{(\infty)}\tau \ln\left(1 - \frac{R}{v_{x_0}\tau}\right) = 0$
In the limit of small resistance, τ is large, that $\frac{R}{v_{x0}\tau} \ll 1$
we use $\ln(1-\epsilon) = -\left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \cdots\right)$

$$\Rightarrow \frac{v_{y_0} + v(\infty)}{v_{x_0}}R - v_{(\infty)}\tau\left[\frac{R}{v_{x_0}\tau} + \frac{1}{2}\left(\frac{R}{v_{x_0}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x_0}\tau}\right)^3\right] = 0$$

$$\Rightarrow \frac{v_{y_0}}{v_{x_0}}\frac{R}{v_{(\infty)}\tau} = \frac{1}{2}\left(\frac{R}{v_{x_0}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x_0}\tau}\right)^3$$

$$\Rightarrow R = \frac{2v_{x_0}v_{y_0}}{g} - \frac{2}{3v_{x_0}\tau}R^2$$

$$\Rightarrow R \approx R_{vac} - \frac{2}{3v_{x_0}\tau}\frac{4v_{x_0}^2v_{y_0}^2}{g^2} = R_{vac}\left[1 - \frac{4}{3}\frac{v_{y_0}}{v_{(\infty)}}\right]$$

when $\frac{v_{y0}}{v(\infty)} \simeq 1$, the effect of air-resistance **canot be neglected**!

Estimations: A metal pellets $D = 0.2$ mm, $v = 1$ m/s at angle 45°.
The range in the absence of resistance:

$R_{vac} = \frac{2v_{x_0}v_{y_0}}{g} = \frac{v^2\sin 2\theta}{g} = \frac{1}{9.8}m \approx 10.2$ cm
For gold $v(\infty) = \frac{\rho D^2 g}{18\eta}$ plug in $\rho = 16$ g/cm^3

$$D = 0.2 \text{ mm}.$$
$$\eta = 1.7 \times 10^{-5} \text{ kg/m} \cdot \text{ s}$$

$\Rightarrow v(\infty) = 21$ m/s
the correction $\frac{4}{3} \cdot \frac{1\times 0.7}{21} \simeq 5\%$

For Al, $\rho = 2.7$ g/cm^3 , which is 1/6 of gold.
Hence, the correction is 6 times larger, $\simeq 30\%$.