Lect 2：Vectors，scalar and cross products，rotations

Outline：

（1）Definition of vector．parallelogram law

（2）inner product

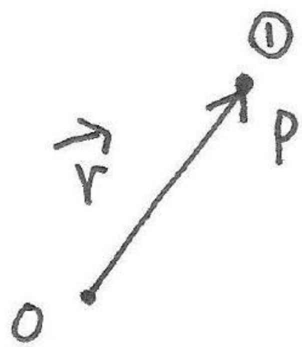
（3）Cross product－directed area

（4）Frame and coordinate transformations

（5）invariance of scalar product．

§ vector：

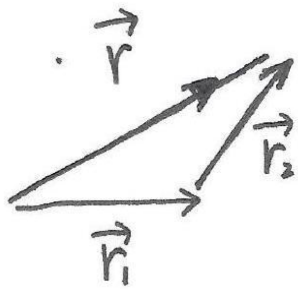
Varies physical quantities have both direction and magnitude which are represented as vectors．For example，the displacement，velocity，acceleration． force，etc，are all vectors．Pictorially，vectors can be represented by a line segment with a direction．



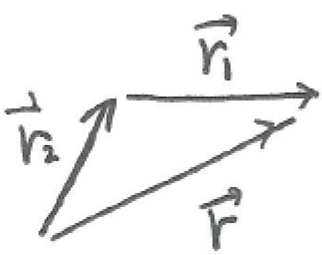
the direction of $\vec{r}$ is often denoted as $\hat{r}$ ． The length of $\hat{r}$ is 1 ，hence，$\hat{r}$ is often called the unit vector．

$-\vec{r}$ has the opposite direction but the same magnitude as $\vec{r}$

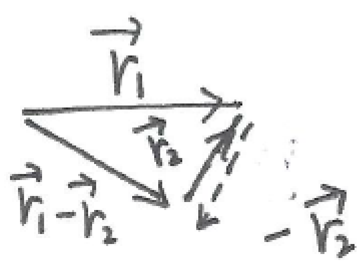
we have $\vec{r}+(-\vec{r})=\vec{r}-\vec{r}=0$.
（3）Summation of two vectors



$\vec{r}=\vec{r}_1+\vec{r}_2$ parallelogram law of addition



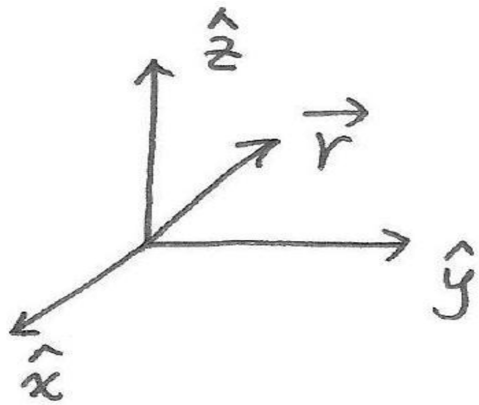
$\vec{r}=\vec{r}_2+\vec{r}_1=\vec{r}_1+\vec{r}_2$



$\vec{r}_1+(-\vec{r}_2)=\vec{r}_1-\vec{r}_2$

（3）Components of a vector

Let $\hat{x}, \hat{y}, \hat{z}$ be a set of orthogonal unit vectors.
They define a cartesian coordinate system.



A vector $\vec{r}$ is written as
$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$， where x, y， and z are called the components. The magnitude of $\vec{r}$ is denoted as $\boldsymbol{r}$ or $|r|$.

$$r = \sqrt{x^2 + y^2 + z^2}.$$

（4）inner product（scalar product）

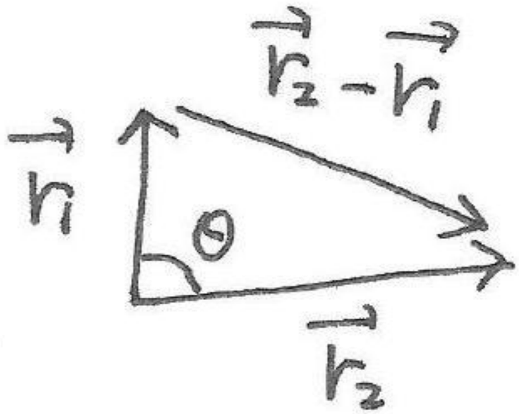
Consider two vectors $\overrightarrow{r_1} = x_1\hat{x} + y_1\hat{y} + z_1\hat{z}$， and $\overrightarrow{r_2} = x_2\hat{x} + y_2\hat{y} + z_2\hat{z}$.
Define their inner product as

$$\overrightarrow{r_1} \cdot \overrightarrow{r_2} = x_1x_2 + y_1y_2 + z_1z_2.$$

then $r^2 = \vec{r} \cdot \vec{r}$

$$\begin{cases} \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \\ \hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{x} = 0 \\ \hat{x} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0 \\ \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{y} = 0 \end{cases}$$

（5）Geometrical meaning of the inner product



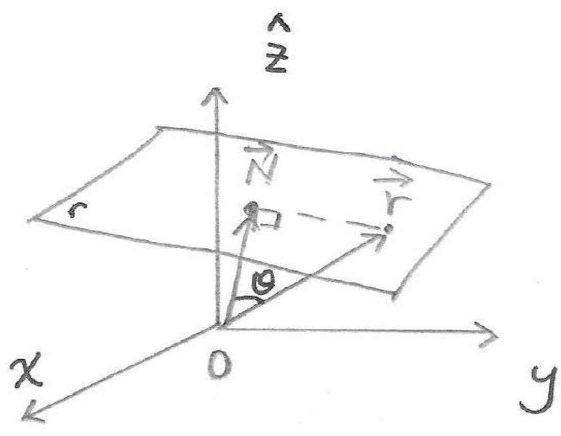
$$|\vec{r}_2 - \vec{r}_1|^2 = (\vec{r}_2 - \vec{r}_1) \cdot (\vec{r}_2 - \vec{r}_1)$$
$$= \vec{r}_2 \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_1 - 2\vec{r}_1 \cdot \vec{r}_2 = r_2^2 + r_1^2 - 2r_1 r_2 \cos\,\theta$$
$$\Rightarrow \vec{r}_1 \cdot \vec{r}_2 = r_1 r_2 \cos\,\theta$$

Applications of the inner product

(1) Equation of a plane

$\overrightarrow{ON}$ is the normal to the plane with the foot N located in the plane.

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ is an arbitary point on the plane.



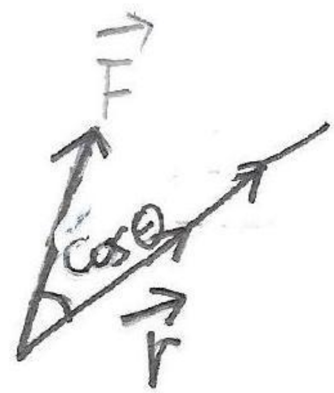
$\vec{r}\cdot\overrightarrow{ON} = r\cdot|ON|\cos\ \theta = |ON|^2$, denote $\overrightarrow{ON} = N_x\hat{x} + N_y\hat{y}$
$+N_z\hat{z}$

$x\cdot N_x + yN_y + zN_z = |ON|^2$

i.e. $\frac{xN_x}{|ON|^2} + \frac{y\cdot N_y}{|ON|^2} + \frac{zN_z}{|ON|^2} = 1.$

(2) Work

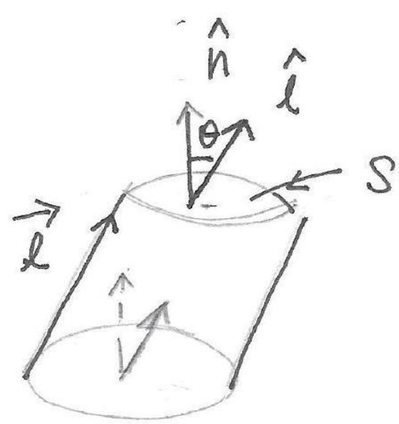
$$W = Fr\cos\ \theta = \vec{F}\cdot\vec{r}$$
$$P = d\omega/dt = \vec{F}\cdot d\vec{r}/dt = \vec{F}\cdot\vec{v}$$



(3) Volume swept: by an area

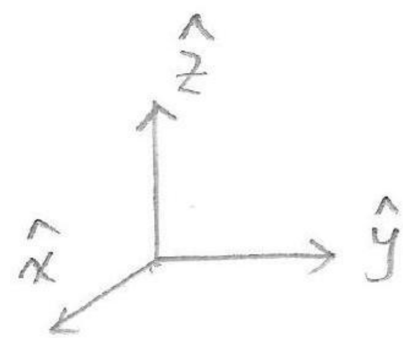
$$\vec{S} = S\hat{n}\ \text{(directed area)}$$

The volume $= \vec{s}\cdot\vec{l} = s\hat{n}\cdot\vec{l}$



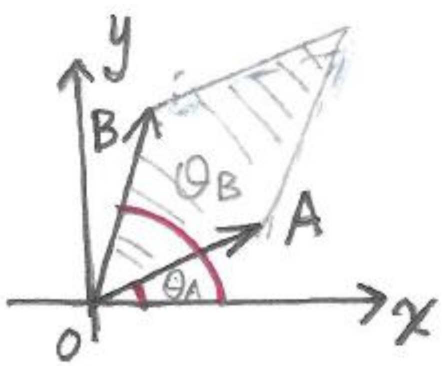
§ cross product:
We define the cross product for the basis vectors as

$$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$$
$$\hat{x} \times \hat{y} = \hat{z}, \qquad \hat{y} \times \hat{z} = \hat{x}, \qquad \hat{z} \times \hat{x} = \hat{y},$$
$$\hat{y} \times \hat{x} = -\hat{z}, \qquad \hat{z} \times \hat{y} = -\hat{x}, \qquad \hat{x} \times \hat{z} = -\hat{y}.$$



Right－hand－thread rule

Consider two vectors $\vec{A} = A_x\hat{x} + A_y\hat{y} \cdot \vec{B} = B_x\hat{x} + B_y\hat{y}$, then



$$\vec{A} \times \vec{B} = \left(A_x\hat{x} + A_y\hat{y}\right) \times \left(B_x\hat{x} + B_y\hat{y}\right) = \left(A_xB_y - A_yB_x\right)\hat{z}$$
$$A_x = |OA|\cos\,\theta_A \qquad A_y = |OA|\sin\,\theta_A$$
$$B_x = |OB|\cos\,\theta_B \qquad B_y = |OB|\sin\,\theta_B$$
$$\vec{A} \times \vec{B} = |OA| \cdot |OB|(\cos\,\theta_A\sin\,\theta_B - \sin\,\theta_A\cos\,\theta_B)\hat{z}$$
$$= |OA| \cdot |OB|\sin\,(\theta_B - \theta_A)\hat{z}$$

Hence, the directed area of a parallegram is $\vec{A} \times \vec{B}$, whose direction is perpendicular to the plane following the right—hand—thread rule.

This conclusion is also true for the general case of vectors $\vec{A}$ and $\vec{B}$.

Assume $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}, \vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$
Then

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(A_yB_z - A_zB_y)$$

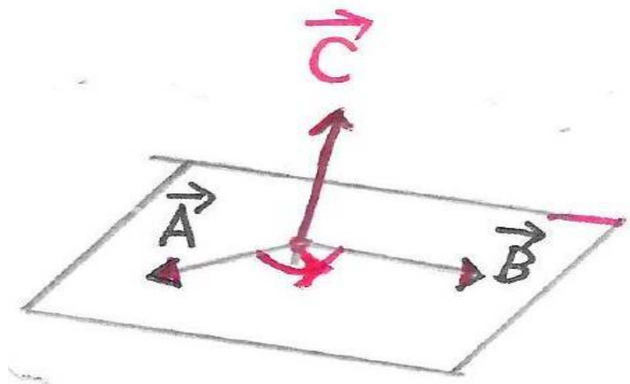
$$+\hat{y}(A_zB_x - A_xB_z)$$
$$+\hat{z}(A_xB_y - A_yB_x)$$

please notice the cyclic (rotation) pattern of indices.

$$\vec{C} \cdot \vec{A} = A_x(A_yB_z - A_zB_y) + A_y(A_zB_x - A_xB_z) + A_z(A_xB_y - A_yB_x) = 0$$

$$\text{or } \vec{C} \cdot \vec{A} = \begin{vmatrix} A_x & A_y & A_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = 0$$

Similarly, $\vec{C} \cdot \vec{B} = 0 \Rightarrow \vec{C} \perp$ plane spanned by $\vec{A}, \vec{B}$
The direction of $\vec{C}$ follows the right-hand rule for the special case discussed for $\vec{A} \cdot \vec{B}$ lying in the xy plane. Since the right or left-hand convention cannot be changed smoothly, the right-hand convention is maintained for the general case.



· $|\boldsymbol{C}|^2$ =?

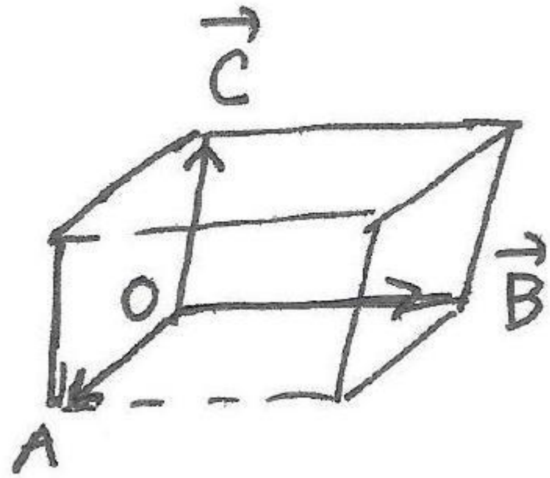
$$|C|^2 = \left(A_yB_z - A_zB_y\right)^2 + (A_zB_x - A_xB_z)^2 + \left(A_xB_y \quad - A_yB_x\right)^2$$
$$= \left(A_x^2 + A_y^2 + A_z^2\right)\left(B_x^2 + B_y^2 + B_z^2\right) - \left(A_xB_x + A_yB_y + A_zB_z\right)^2$$
$$= |A|^2|B|^2 - |A \cdot B|^2 = |A|^2|B|^2(1 - \cos^2\ \theta) = |A|^2|B|^2\sin^2\ \theta$$

Hence $\vec{C}$ is the direted area of the parallagram $\vec{A} \times \vec{B}$ for the general case!

It's easy to check that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$. The direction of the area $\vec{B} \times \vec{A}$ is opposite to that of $\vec{A} \times \vec{B}$.

Volume of a parallel piped

$$\vec{S}_{AB} \cdot \overrightarrow{OC} = (\overrightarrow{OA} \times \overrightarrow{OB}) \cdot (\overrightarrow{OC})$$



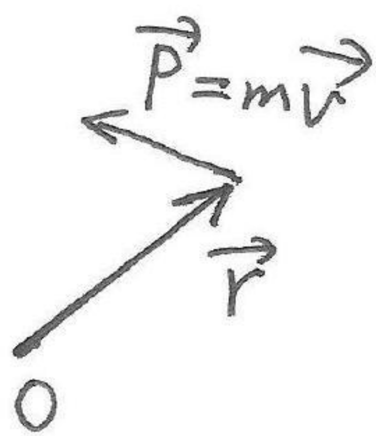
or simply $(\vec{A} \times \vec{B}) \cdot \vec{C}$.
We could also interpret this volume as $\vec{S}_{BC} \cdot \overrightarrow{OA} = (\vec{B} \times \vec{C}) \cdot \vec{A}$
and $\vec{S}_{CA} \cdot \overrightarrow{OB} = (\vec{C} \times \vec{A}) \cdot \vec{B}$.
The scalar triple is invariant under cyclically permutation.

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B}$$
$$= -(\vec{B} \times \vec{A})\vec{C} = -(\vec{C} \times \vec{B}) \cdot \vec{A} = -(\vec{A} \times \vec{C}) \cdot \vec{B}$$

− 'sign appears for exchanging two vectors.

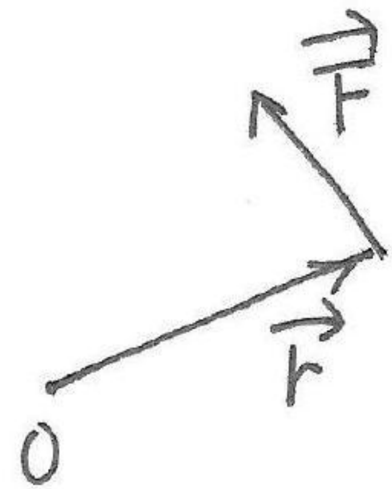
examples:
（1）angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$$



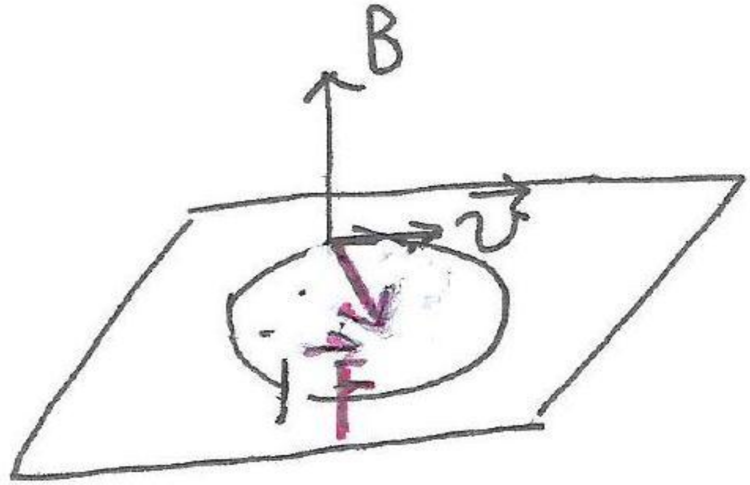
（3）torque

$$\vec{N} = \vec{r} \times \vec{F}$$



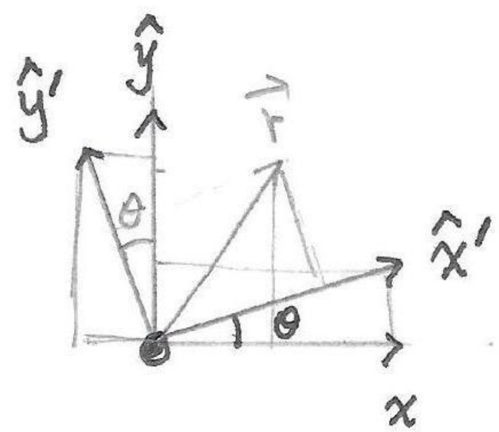
（3）Lerentz force

$$\vec{F} = \frac{q}{c}\vec{v} \times \vec{B} \text{ (Gaussian)}$$
$$= q\vec{v} \times \vec{B} \ (SI)$$



§ Frame transformation -- passive view

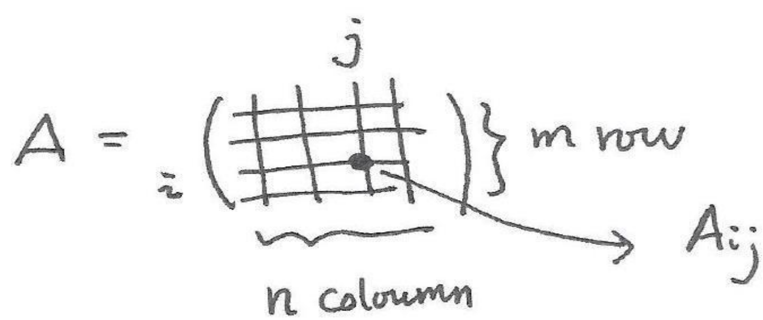
Consider two frames with the base vectors $(\hat{x}', \hat{y}', \hat{z}')$ and $(\hat{x}, \hat{y}, \hat{z})$ with $\hat{z} = \hat{z}'$,



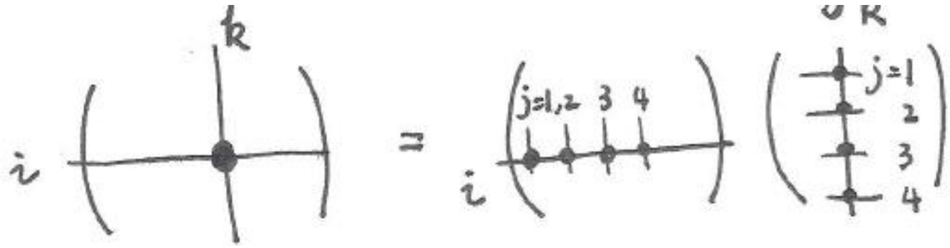
but $\hat{x}, \hat{y}$ and $\hat{x}', \hat{y}'$ are rotated at the angle of θ .

$$\begin{aligned} \hat{x}' &= \hat{x}\cos\,\theta + \hat{y}\sin\,\theta \\ \hat{y}' &= -\hat{x}\sin\,\theta + \hat{y}\cos\,\theta \end{aligned} \Rightarrow (\hat{x}', \hat{y}') = (\hat{x}, \hat{y}) \begin{pmatrix} \cos\,\theta & -\sin\,\theta \\ \sin\,\theta & \cos\,\theta \end{pmatrix}$$

Matrix:



matrix product $C = AB$



$$C_{ik} = \sum_j A_{ij}B_{jk}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg, & af+bh \\ ce+dg, & cf+dh \end{pmatrix}$$

$$\neq$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+fc, & eb+df \\ ag+hc, & bg+dh \end{pmatrix}$$

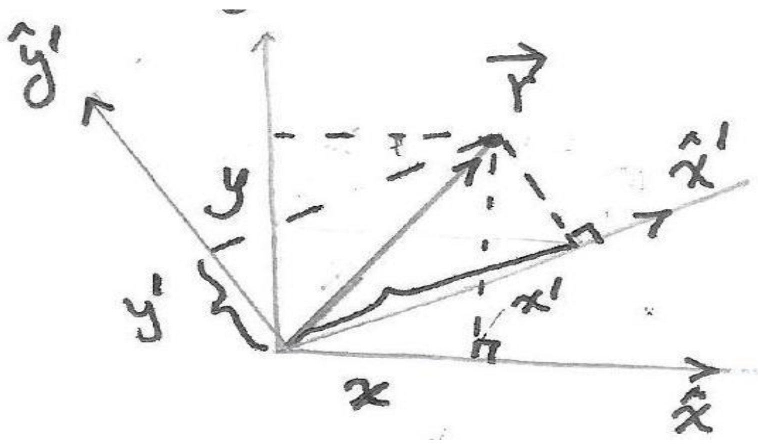
$(A^{\mathrm{T}})_{ij} = A_{ji}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $T \rightarrow$ transpose

Coordinate transformation：

$$\vec{r} = x\hat{x} + y\hat{y}$$
$$= x'\hat{x}' + y'\hat{y}'$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$(x', y') = (x, y)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

⋆ Invariance of inner product



$\overrightarrow{r_1} = \hat{x}x_1 + \hat{y}y_1, \overrightarrow{r_2} = \hat{x}x_2 + \hat{y}y_2 \Rightarrow$ unden the basis$(\hat{x}, \hat{y})$

we have $\vec{r}_1 \cdot \vec{r}_2 = x_1x_2 + y_1y_2 = (x_1 \quad y_1)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

Similarly under the basis of $(\hat{x}', \hat{y}')$, we urite

$$\vec{r}_1 = \hat{x}'x_2' + \hat{y}'y_1', \vec{r}_2 = \hat{x}'x_2' + \hat{y}'y_2' \Rightarrow \vec{r}_1 \cdot \vec{r}_2 = (x_1', y_1')\begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$$

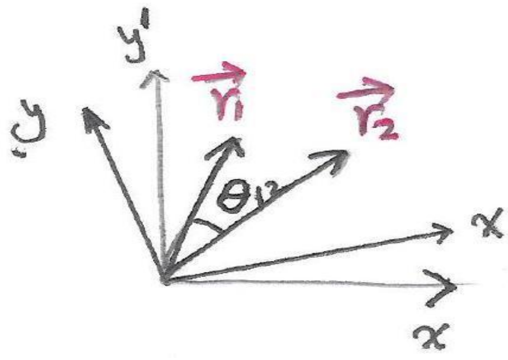
Are they consistent? Yes, otherwise it does not look good.

Proof: $(x_1' \quad y_1')\begin{pmatrix} x_2' \\ y_2' \end{pmatrix} =$

$(x_1 \quad y_1)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$$= (x_1 \quad y_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (x_1 \quad y_1)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

This makes sense that $\vec{r}_1 \cdot \vec{r}_2 = |r_1||r_2| \cos\theta_{12}$, which should be independent of frame transformation, and this is why



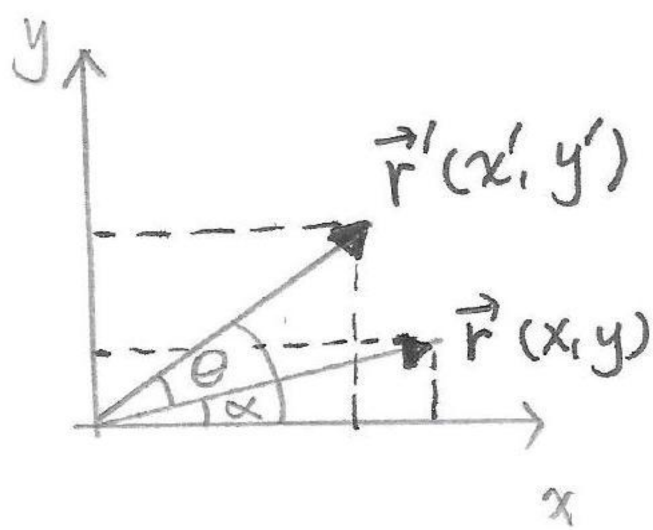
$\vec{r}_1 \cdot \vec{r}_2$ is called the scalor product：

scalar：a quantity is invariant under the frame transformation：

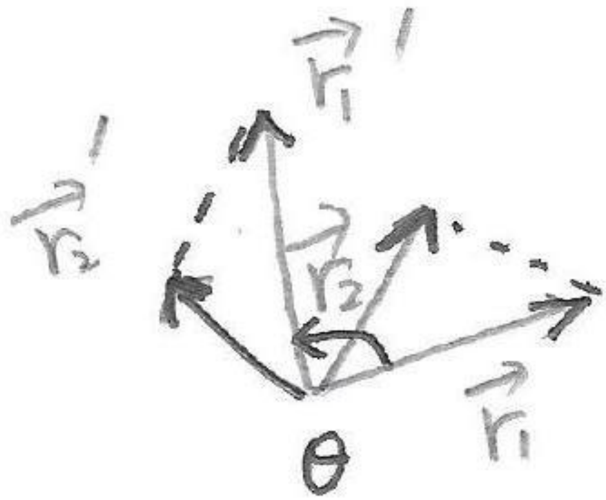
—Initiative viewpoint：fix frame but rotate the vector

$$
\begin{aligned}
x' &= r\cos(\theta+\alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\
&= \cos\theta\, x - \sin\theta\, y \\
y' &= r\sin(\theta+\alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \\
&= \sin\theta\, x + \cos\theta\, y
\end{aligned}
$$

$$
\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
$$



consider a pair of vectors $\vec{r}_1, \vec{r}_2$ rotate around the $\hat{z}-$axis at the angle of θ



we arrive at $\vec{r}_1', \vec{r}_2'$, then it′ s easy to show $\vec{r}_1' \cdot \vec{r}_2' = \vec{r}_1 \cdot \vec{r}_2$. We denote transformation matrix $U = \begin{pmatrix} \cos\ \theta & -\sin\ \theta \\ \sin\ \theta & \cos\ \theta \end{pmatrix}$

$$\vec{r}' \cdot \vec{r}_2' = (x_1', y_1') \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = (x_1 y_1) u^\top u \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

please check $u^\top u = I$, which we call $u^\top and\ u$ orthogonal matrix.