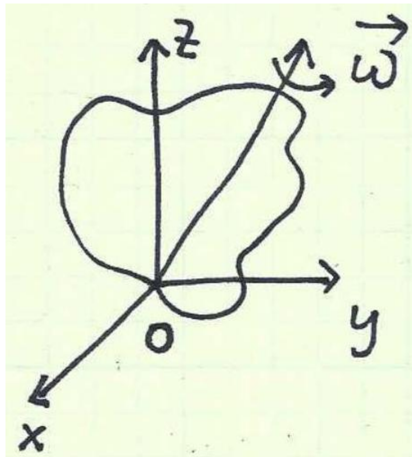
Lect 14 Rigid body (II) - Euler equation

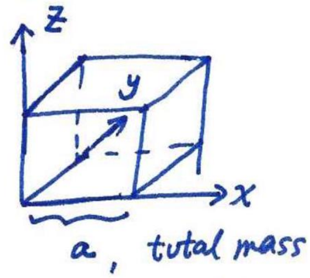
Definition: moment of inertia, rotation around any axis



$$\vec{\omega} = (\omega_x,\ \omega_y,\ \omega_z)$$
$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$
$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}\, r^2 - \vec{r}\,(\vec{\omega}\cdot\vec{r}) = (\omega_x r^2,\ \omega_y r^2,\ \omega_z r^2) - (\vec{\omega}\cdot\vec{r})(x,\ y,\ z)$$
$$= \Big[(y^2+z^2)\,\omega_x - xy\,\omega_y - xz\,\omega_z,\quad - yx\,\omega_x + (z^2+x^2)\,\omega_y - yz\,\omega_z,\quad - zx\,\omega_x - zy\,\omega_y + (x^2+y^2)\,\omega_z\Big]$$
$$\Rightarrow \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$
$$\Rightarrow \quad L_a = I_{ab}\,\omega_b$$
where $I_{ab} = \delta_{ab}\left(\sum_{\alpha=1}^{N} m_\alpha r_\alpha^2\right) - \sum_{\alpha=1}^{N} m_\alpha x_a x_b$
$$= \int dx\,dy\,dz\ \rho\left[r^2\delta_{ab} - x_a x_b\right] \qquad \Leftarrow\ I_{ab} = I_{ba}\ \text{symmetric tensor.}$$

Example: inertia tensor of a solid cube, rotating

(1) Around the corner (take the cube of side a with one corner at the origin, uniform density ρ)



$$I_{xx} = \int_0^a dx \int_0^a dy \int_0^a dz\, \rho\,(y^2+z^2) = \rho \int_0^a dx \int_0^a dy \int_0^a dz\,(y^2+z^2)$$
$$= \rho \left[\int_0^a dx\right] \left[\int_0^a dy\, y^2 \int_0^a dz + \int_0^a dz\, z^2 \int_0^a dy\right]$$
$$= \rho \left[a\left(\frac{a^3}{3}\cdot a\right) + a\left(\frac{a^3}{3}\cdot a\right)\right] = \rho\, a \left(\frac{a^4}{3}+\frac{a^4}{3}\right) = \rho\, a \cdot \frac{2a^4}{3} = \frac{2}{3}\rho a^5 = \frac{2a^2}{3} M\,,$$

where $M = \rho a^3$ is the total mass. By symmetry,

$$I_{yy} = I_{zz} = I_{xx}\,.$$

For the products of inertia (off-diagonal terms),

$$I_{xy} = -\int_0^a dx \int_0^a dy \int_0^a dz\, \rho\, xy = -\rho \left(\int_0^a dx\, x\right)\left(\int_0^a dy\, y\right)\left(\int_0^a dz\right)$$
$$= -\rho\left(\frac{a^2}{2}\right)\left(\frac{a^2}{2}\right)(a) = -\rho\,\frac{a^5}{4} = -\frac{a^2}{4}M\,.$$

Similarly $I_{yz} = I_{zx} = I_{xy}$ by symmetry for this corner choice. Therefore,

$$I = Ma^2 \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}.$$

If the cube rotates around the z-axis, we take $\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$

$$\Rightarrow \vec{L} = Ma^2\omega \begin{bmatrix} -1/4 \\ -1/4 \\ 2/3 \end{bmatrix}.$$

(2) If the cube rotates around its center:
Now the origin is at the center of the cube, so $x, y, z \in [-a/2, a/2]$.

$$
\begin{aligned}
I_{xx} &= \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz\, \rho\,(y^2+z^2) \\
&= \rho\left[\left(\int_{-a/2}^{a/2} dx\right)\left(\int_{-a/2}^{a/2} dy\, y^2 \int_{-a/2}^{a/2} dz + \int_{-a/2}^{a/2} dz\, z^2 \int_{-a/2}^{a/2} dy\right)\right] \\
&= \rho\left[a\left(\frac{2}{3}\left(\frac{a}{2}\right)^3 \cdot a\right)\right] = \rho\, a\left[\frac{2}{3}\frac{a^3}{8}\cdot a\right] \times 2 = \rho\, a \cdot \frac{a^4}{12} \times 2 = \rho\,\frac{a^5}{6} = \frac{a^2}{6} M\,,
\end{aligned}
$$

so

$$
I_{xx} = I_{yy} = I_{zz} = \frac{Ma^2}{6}\,, \qquad I_{xy} = I_{yz} = I_{zx} = 0\,,
$$

and therefore

$$
I = \frac{Ma^2}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \vec{L} = \frac{Ma^2}{6}\,\vec{\omega}\,.
$$

The inertia tensor I depends on the choice of origin and the orientation of the axes.

Kinetic energy of fixed-point rotation

$$
E_k = \frac{1}{2}\sum_\alpha m_\alpha\, \vec{v}_\alpha^2\,.
$$

For rigid rotation,

$$
\vec{v} = \vec{\omega}\times\vec{r} \quad \Rightarrow \quad v^2 = \omega^2 r^2 - (\vec{\omega}\cdot\vec{r})^2\,.
$$

Thus

$$
E_k = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha\Big[\omega_x^2(y^2+z^2)+\omega_y^2(z^2+x^2)+\omega_z^2(x^2+y^2)-2\omega_x\omega_y xy-2\omega_y\omega_z yz-2\omega_z\omega_x zx\Big]\,.
$$

In matrix form,

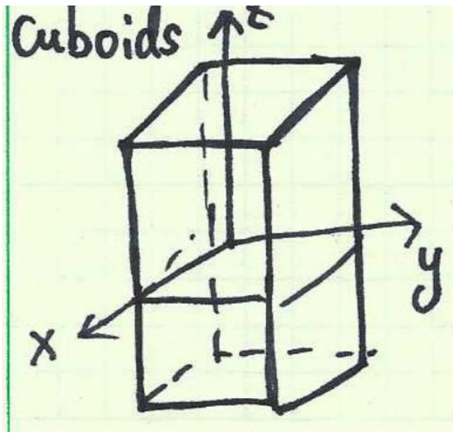
$$
E_k = \frac{1}{2}\begin{bmatrix}\omega_x & \omega_y & \omega_z\end{bmatrix}\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{bmatrix} = \frac{1}{2}\,\omega_a I_{ab}\omega_b = \frac{1}{2}\,\vec{\omega}\cdot\vec{L}\,.
$$

I_{ab} is a 3×3 symmetric real matrix. It can be diagonalized by an orthogonal matrix. Suppose its diagonal form is $I_{ab}\to \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$ in the principal-axis frame. Then

$$
E_k = \frac{1}{2}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right),
$$

where $(\omega_1,\omega_2,\omega_3)$ is the projection of $\vec{\omega}$ along the principal axes.

Example: a cuboid aligned with the x, y, z axes:

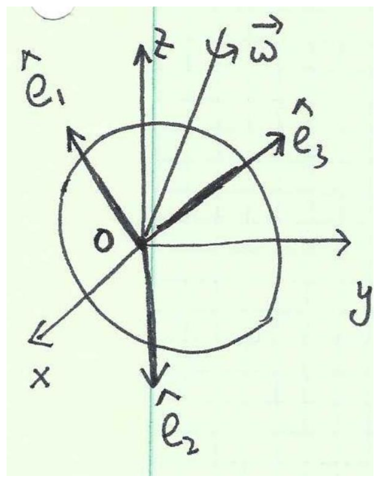


$$I_{xy} = I_{yz} = I_{zx} = 0, \qquad I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}.$$

The x, y, z axes are principal axes.

Euler equation

Space frame: an inertial frame with fixed x, y, z axes.
Body frame: the principal axes (body axes) $\hat{e}_1, \hat{e}_2, \hat{e}_3$,
and the moments of inertia along them are $\lambda_1, \lambda_2, \lambda_3$, respectively.
Rotating frame / non-inertial frame.
If the body has a fixed point, we set the origin O at that point. Then we do not need to worry about the (unknown) force exerted by the support at O. If the body has no fixed point, we choose the origin O at the center of mass.



If the rigid body rotates with angular velocity $\vec{\omega}$, we can write in the body frame

$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3,$$

and

$$\vec{L} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3.$$

The torque is $\vec{\Gamma}$:

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{space}} = \vec{\Gamma} = \Gamma_1 \hat{e}_1 + \Gamma_2 \hat{e}_2 + \Gamma_3 \hat{e}_3 \,.$$

$$\boxed{\tfrac{d\hat{e}_i}{dt} = \vec{\omega} \times \hat{e}_i}$$

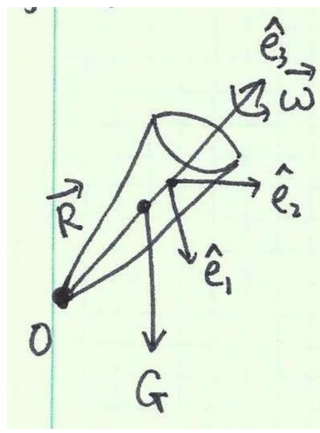
since in the body frame, the basis vectors $\hat{e}_i$ rotate with angular velocity $\vec{\omega}$.

$$\begin{aligned} &\lambda_1 \dot{\omega}_1 \hat{e}_1 + \lambda_2 \dot{\omega}_2 \hat{e}_2 + \lambda_3 \dot{\omega}_3 \hat{e}_3 + \vec{\omega} \times (\lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3) \\ &= \left(\frac{d\vec{L}}{dt}\right)_{\text{space}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{L} \,. \end{aligned}$$

Hence, component-wise (Euler's equations):

$$\begin{aligned}
\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\,\omega_2\omega_3 &= \Gamma_1, \\
\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3\omega_1 &= \Gamma_2, \\
\lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\,\omega_1\omega_2 &= \Gamma_3.
\end{aligned}$$

Application: In many rigid-body problems, $\Gamma_{1,2,3}$ are components in the rotating (body) frame, which can make them awkward to track. Euler's equations are especially useful for $\vec{\Gamma} = 0$, or for the case where one component is zero.



Symmetric top spinning in a gravitational field.

$$\vec{\Gamma} = \vec{R} \times \vec{G} = R\hat{e}_3 \times \vec{G} \Rightarrow \vec{\Gamma} \cdot \hat{e}_3 = 0.$$

For a symmetric top, $\lambda_1 = \lambda_2 \equiv \lambda$. Because gravity produces no torque about $\hat{e}_3$ through the contact point,

$$\lambda_3\dot{\omega}_3 = 0\,,$$

so ω_3 is constant.

Euler equation with zero torque

If $\vec{\Gamma} = 0$, then $\vec{L}$ is conserved in the lab (space) frame, but not necessarily in the body frame:

$$\begin{aligned}
\lambda_1\dot{\omega}_1 &= (\lambda_2 - \lambda_3)\,\omega_2\omega_3, \\
\lambda_2\dot{\omega}_2 &= (\lambda_3 - \lambda_1)\,\omega_3\omega_1, \\
\lambda_3\dot{\omega}_3 &= (\lambda_1 - \lambda_2)\,\omega_1\omega_2.
\end{aligned}$$

$\star$ If we start with $\vec{\omega}$ along a principal axis, say $\hat{e}_1$:

$$\omega_2 = \omega_3 = 0 \text{ at } t = 0 \quad \Rightarrow \quad \begin{cases} \lambda_1\dot{\omega}_1 = 0, \\ \lambda_2\dot{\omega}_2 = 0, \\ \lambda_3\dot{\omega}_3 = 0, \end{cases}$$

so $\vec{\omega}$ is constant (pure spin about that axis).

$\star$ If $\vec{\omega}$ is not along any principal axis and $\lambda_1 \neq \lambda_2 \neq \lambda_3$, then at most only one of $\omega_{1,2,3}$ can remain nonzero. In general we have $\dot{\vec{\omega}} \neq 0$. Thus although $\vec{L}$ is conserved, $\vec{\omega}$ is not.

Stability of rotation around a principal axis

Suppose at $t = 0$, $\vec{\omega} = \omega_3 \hat{e}_3$ with $\omega_1 = \omega_2 = 0$. Now give a small perturbation in ω_1, ω_2. Will $\omega_{1,2}$ grow? From Euler's equations,

$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\,\omega_1 \omega_2 \sim \text{2nd-order small} \approx 0,$$

so $\omega_3 \approx$ const.

Linearize:

$$\begin{cases} \dot{\omega}_1 = \dfrac{\lambda_2 - \lambda_3}{\lambda_1}\,\omega_3\,\omega_2, \\ \dot{\omega}_2 = \dfrac{\lambda_3 - \lambda_1}{\lambda_2}\,\omega_3\,\omega_1. \end{cases}$$

Try solutions $\omega_1(t) = Ae^{\mu t}$, $\omega_2(t) = Be^{\mu t}$. Then

$$\begin{cases} A\mu = \dfrac{\lambda_2 - \lambda_3}{\lambda_1}\,\omega_3\,B, \\ B\mu = \dfrac{\lambda_3 - \lambda_1}{\lambda_2}\,\omega_3\,A, \end{cases} \qquad \Rightarrow \qquad \mu^2 = \frac{\lambda_2 - \lambda_3}{\lambda_1}\frac{\lambda_3 - \lambda_1}{\lambda_2}\,\omega_3^2.$$

Case (1): If λ_3 lies between λ_1 and λ_2, then $\mu^2 > 0$. We get a pair of real solutions $\mu_{1,2} = \pm\mu_0$ with

$$\mu_0 = \omega_3 \sqrt{\frac{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2}} > 0.$$

Then

$$\omega_1(t) = Ae^{\mu_0 t} + A' e^{-\mu_0 t}, \qquad \omega_2(t) = Be^{\mu_0 t} + B' e^{-\mu_0 t},$$

and $\omega_{1,2}(t)$ have exponentially growing pieces $\sim e^{\mu_0 t}$. The rotation is **unstable**.

Case (2): If λ_3 is either the largest or the smallest among $\lambda_1, \lambda_2, \lambda_3$, then $\mu = \pm i\mu_0$ with

$$\mu_0 = \frac{\omega_3}{\sqrt{\lambda_1 \lambda_2}} \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\,.$$

Write

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\mu_0 t},$$

which implies

$$A\, i\mu_0 = \frac{\lambda_2 - \lambda_3}{\lambda_1}\,\omega_3\,B \quad \Rightarrow \quad B = \frac{i}{\sqrt{\lambda_1 \lambda_2}} \frac{\lambda_1}{\lambda_2 - \lambda_3} \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\,.$$

Hence

$$\begin{pmatrix}\omega_1(t)\\ \omega_2(t)\end{pmatrix} = A\begin{pmatrix}1\\ \pm i\sqrt{\dfrac{\lambda_1(\lambda_1-\lambda_3)}{\lambda_2(\lambda_2-\lambda_3)}}\end{pmatrix}e^{i\mu_0 t},$$

where the sign depends on whether $\lambda_2 > \lambda_3$ (+) or $\lambda_2 < \lambda_3$ (-).

The complex conjugate gives the other solution with $e^{-i\mu_0 t}$. Taking a real linear combination,

$$\begin{pmatrix}\omega_1(t)\\ \omega_2(t)\end{pmatrix} = A'\begin{pmatrix}\cos(\mu_0 t+\varphi)\\ \mp\sin(\mu_0 t+\varphi)\sqrt{\frac{\lambda_1}{\lambda_2}\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}}\end{pmatrix},$$

which is purely oscillatory ($\lambda_2 > \lambda_3 \rightarrow$"-"). This rotation is **stable**.

Symmetric top ($\lambda_1 = \lambda_2 = \lambda$)

For a symmetric top, let $\lambda_1 = \lambda_2 = \lambda$ and λ_3 possibly different. Then $\omega_3 =$ const.

A standard solution is

$$\begin{pmatrix}\omega_1(t)\\ \omega_2(t)\\ \omega_3\end{pmatrix} = \begin{pmatrix}\omega_0\cos\left(\dfrac{\omega_3(\lambda-\lambda_3)}{\lambda}t\right)\\ -\omega_0\sin\left(\dfrac{\omega_3(\lambda-\lambda_3)}{\lambda}t\right)\\ \omega_3\end{pmatrix}.$$

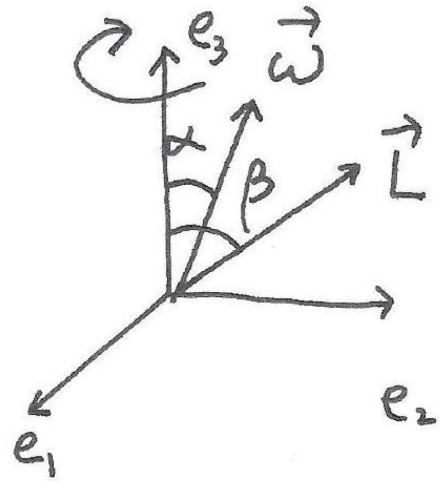
For an "egg-shaped" (prolate) configuration with $\lambda_3 < \lambda = \lambda_1 = \lambda_2$, we can write

$$\vec{\omega} = \omega_0\cos(\Omega_b t)\,\hat{e}_1 - \omega_0\sin(\Omega_b t)\,\hat{e}_2 + \omega_3\hat{e}_3, \quad \Omega_b = \frac{\omega_3}{\lambda}(\lambda-\lambda_3).$$

In the body frame, $\vec{\omega}$ precesses around $-\hat{e}_3$.

The angular momentum is

$$\vec{L} = \lambda\,\omega_0\cos(\Omega_b t)\,\hat{e}_1 - \lambda\,\omega_0\sin(\Omega_b t)\,\hat{e}_2 + \lambda_3\omega_3\hat{e}_3.$$



Define the tilt angles:

$$\tan\alpha = \frac{\omega_0}{\omega_3}, \qquad \tan\beta = \frac{\omega_0}{\omega_3}\frac{\lambda}{\lambda_3} \quad \Rightarrow \quad \beta > \alpha.$$

$\vec{\omega}$ and $\vec{L}$ both precess around $-\hat{e}_3$ with the same angular velocity Ω_b.