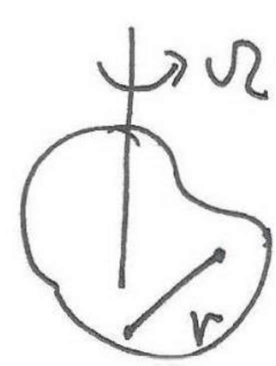
Lect 13 Rigid body (I) - fixed axis rotation

- Rotation around a fixed axis
- Moment of inertia
- equation of motion of fixed axis rotation

What's rigid body?



No elasticity!
The distance between any two points is fixed during the motion, i.e. no shape deformation.

How many degrees of freedom to describe a rigid body motion? let's check

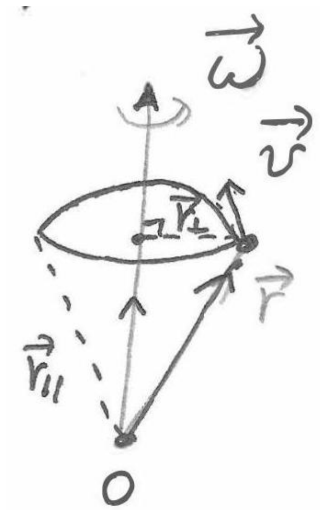
# of particles	# of degrees of freedom	if all dist fixed
1	3	point
2	$2 \times 3 - 1 = 5$	line segment
3	$3 \times 3 - 3 = 6$	triangle
4	no extra degrees of freedom	tetrahedron
$\vdots$	6	
N	6	

3 translational degrees of freedom
3 rotational degrees of freedom

Laws of rotation around a fixed axis

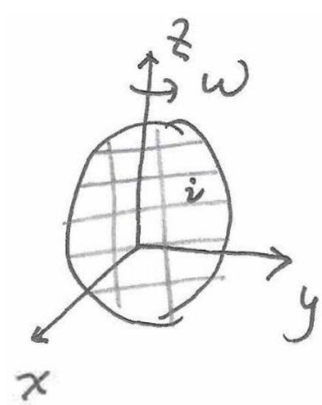
Instantaneous velocity at a point: write $\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$. $\vec{r}_{\parallel}$ is the component parallel to the rotation axis, $\vec{r}_{\perp}$ is the component perpendicular to the axis.

$$\vec{v} = \vec{\omega} \times \vec{r}_{\perp} = \vec{\omega} \times \left(\vec{r}_{\perp} + \vec{r}_{\parallel}\right) = \vec{\omega} \times \vec{r}$$



The angular momentum

$$\vec{L} = \sum_{i=1}^{N} \vec{l}_i = \sum_{i=1}^{N} m_i \, \vec{r}_i \times \vec{v}_i$$
$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$
$$\Rightarrow \quad \vec{L} = \sum_{i=1}^{N} m_i \, \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$



in our case

$$\vec{\omega} = \omega\,\hat{z}, \qquad \vec{r}_i = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}$$

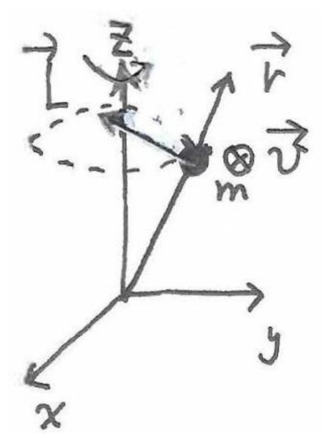
$$\begin{aligned}\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) &= \vec{\omega}\, r_i^2 - \vec{r}_i\, (\vec{\omega} \cdot \vec{r}_i) \\ &= \left(x_i^2 + y_i^2 + z_i^2\right) \omega\,\hat{z} - \omega \left(z_i x_i \hat{x} + z_i y_i \hat{y} + z_i^2 \hat{z}\right) \\ &= -\omega z_i x_i \hat{x} - \omega z_i y_i \hat{y} + \omega \left(x_i^2 + y_i^2\right) \hat{z}\end{aligned}$$

$$L_z = \sum_{i=1}^{N} m_i\,\omega\, \left(x_i^2 + y_i^2\right) = I_z\,\omega, \quad \text{where } I_z = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

$\longrightarrow I_z = \int dx\,dy\,dz\ \rho \left(x^2 + y^2\right) \longleftarrow$ moment of inertia

$$\left.\begin{aligned} L_x &= -\left(\textstyle\sum_i m_i x_i z_i\right) \omega = I_{xz}\,\omega \neq 0 \\ L_y &= -\left(\textstyle\sum_i m_i y_i z_i\right) \omega = I_{yz}\,\omega \neq 0 \end{aligned}\right\} \quad \text{in general}$$

Example: If the mass distribution is unsymmetric with respect to the rotation axis, $\vec{L} \not\parallel \vec{\omega}$.



Kinetic energy

$$E_k = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i\,\omega^2 \left(x_i^2 + y_i^2\right) = \frac{I_z}{2}\,\omega^2$$

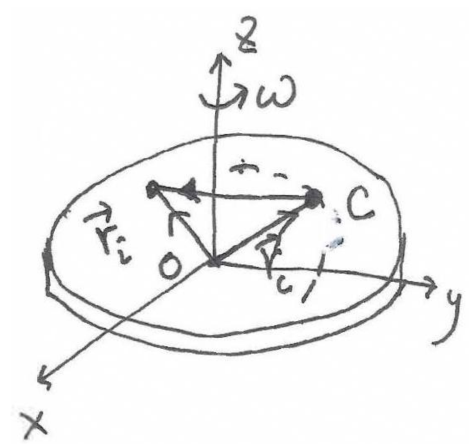
$$\vec{v}_i = -\omega y_i \hat{x} + \omega x_i \hat{y}$$

Parallel axis theorem

If the rotation axis does not pass the center of mass, denote the distance between the mass center and the rotation axis as r_c, then

$$I_z = I_{cz} + M r_c^2,$$

where I_{cz} is the moment of inertia if the rotation axis passes the center of mass.



Write

$$\vec{r}_i = \vec{r}_c + \vec{r}_i', \qquad \vec{r}_i' \text{ is the displacement of "}i\text{th point" relative to } C$$

$$\begin{aligned} I_z &= \sum_i m_i \left|\vec{r}_c + \vec{r}_i'\right|^2 = \sum_i m_i \left(r_c^2 + r_i'^2 + 2\vec{r}_c \cdot \vec{r}_i'\right) \\ &= M r_c^2 + I_{cz} + 2\vec{r}_c \cdot \sum_i m_i \vec{r}_i' \end{aligned}$$

by definition

$$\sum_i m_i \vec{r}_i = \left(\sum_i m_i\right) \vec{r}_c \quad \Rightarrow \quad \sum_i m_i \vec{r}_i' = 0$$

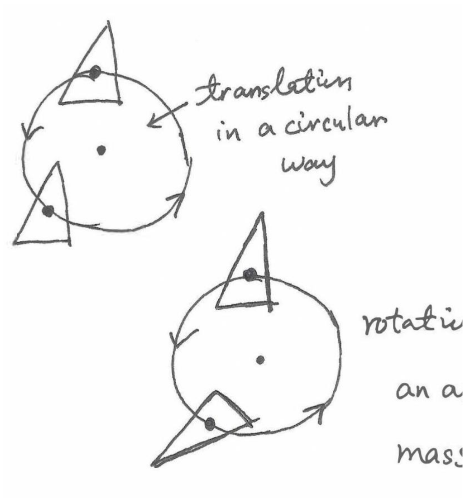
Hence

$$I_z = I_{cz} + M r_c^2.$$

Then

$$\mathbb{E}_k = \frac{1}{2} I_z \omega^2 = \frac{1}{2} I_{cz} \omega^2 + \frac{1}{2} M \left(r_c \omega\right)^2.$$

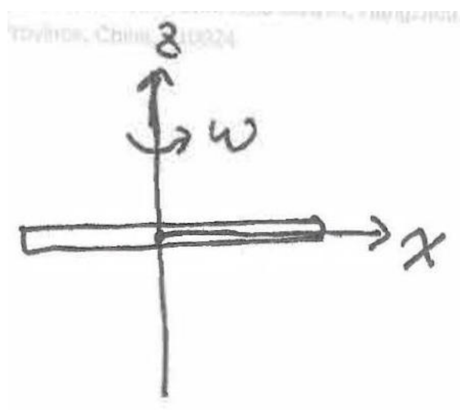
Kinetic energy of a rotating slab = kinetic energy in the center of mass frame + the translation energy as a whole object



rotation around an axis away from the center of mass.

Examples of moment of inertia

(1) Uniform thin rod - rotation axis passes center of mass

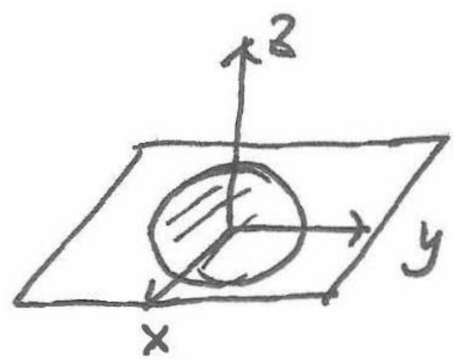


$$I_c = \int_{-L/2}^{L/2} dx\, \rho x^2 = \frac{\rho}{3} x^3 \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{\rho}{12} L^3 = \frac{(\rho L) L^2}{12} = \frac{M}{12} L^2$$

if the rotation axis is located at one end, then

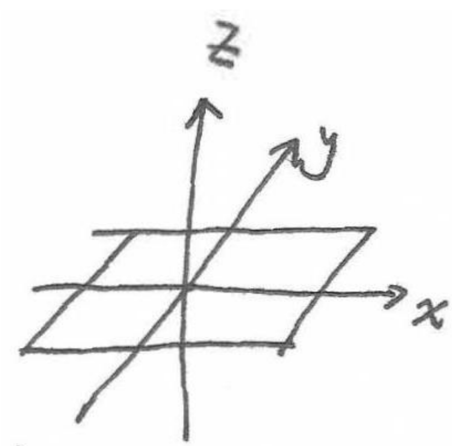
$$I_z = I_{cz} + M \left(\frac{L}{2}\right)^2 = \frac{M}{3} L^2$$

(2) Circular disk - rotation axis passes its center



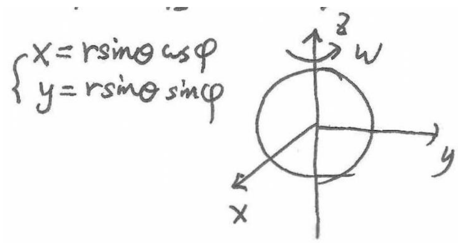
$$
\begin{aligned}
I_z &= \int dx \int dy \; \rho\left(x^2+y^2\right) = \rho \int_0^R r\,dr \int_0^{2\pi} d\theta \; r^2 \\
&= 2\pi\rho \cdot \frac{R^4}{4} = \left(\pi R^2 \rho\right)\frac{R^2}{2} = \frac{1}{2} M R^2
\end{aligned}
$$

(3) rectangular plate - axis passing the center



$$
\begin{aligned}
I_z &= \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \; \rho\left(x^2+y^2\right) \\
&= \rho\left(b \cdot \frac{x^3}{3}\bigg|_{-\frac{a}{2}}^{\frac{a}{2}} + a \cdot \frac{y^3}{3}\bigg|_{-\frac{b}{2}}^{\frac{b}{2}}\right) \\
&= \rho\left(\frac{ba^3}{12} + \frac{ab^3}{12}\right) = \frac{\rho ab}{12}\left(a^2+b^2\right) = \frac{M}{12}\left(a^2+b^2\right)
\end{aligned}
$$

(4) uniform solid sphere



$$I = \int r^2\,dr\,\sin\theta\,d\theta\,d\varphi\,\rho\left(r^2\sin^2\theta\right)$$

$$= \rho\int_0^R r^4 dr \int_0^\pi \sin^3\theta\,d\theta \int_0^{2\pi} d\varphi$$

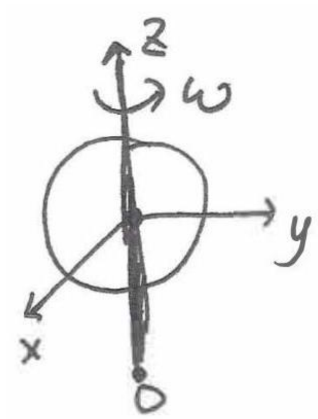
$$\int_0^\pi \sin^3\theta\,d\theta = -\int_0^\pi \sin^2\theta\,d(\cos\theta) = \int_{-1}^{1}(1-x^2)dx$$

$$= \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}$$

$$\Rightarrow I = \rho\cdot\frac{R^5}{5}\cdot\frac{4}{3}\cdot 2\pi = \left(\frac{4}{3}\pi R^3\rho\right)\frac{2}{5}R^2 = \frac{2}{5}MR^2$$

Equation of motion for fixed axis rotation

Set $\vec{\omega}$ is along the $\hat{z}$-axis, for fixed axis rotation



Let's prove

$$\frac{dL_z}{dt} = \tau_z \quad \leftarrow \text{ torque}$$

Actually, we can pick up any point O at the rotation axis as the origin. We proved before

$$\frac{d}{dt}\vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{\text{ex}}$$

we take the z-component:

(1) $L_z = I_z \omega$ and

$$I_z = \int dx\, dy\, dz\ \rho \left(x^2 + y^2\right)$$

i.e. L_z does not depend on the location of "O" on the axis.

(2)

$$\left(\vec{r}_i \times \vec{F}_i^{\text{ex}}\right)_z = x_i F_i^y - y_i F_i^x$$

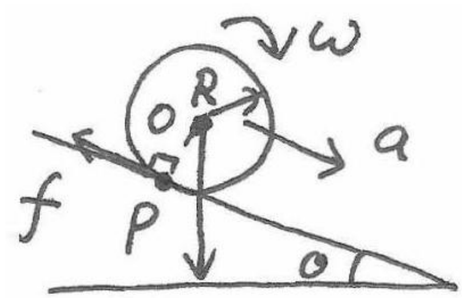
which does not depend on the z-location of "O" either. Hence, we have

$$\boxed{\frac{dL_z}{dt} = \tau_z}$$

Example: rolling without slipping

(1) Constraint:

$$\vec{v}_p = \vec{v}_0 + \vec{\omega} \times \vec{r}_{op} = 0$$



$$\Rightarrow v_0 = \omega R \Rightarrow a = R\frac{d\omega}{dt}$$

with respect to center of mass:

$$f \cdot R = I\frac{d\omega}{dt}$$

$$Mg\sin\theta - f = Ma$$

$$\Rightarrow Mg\sin\theta - I\frac{d\omega}{dt} \cdot \frac{1}{R} = Ma$$

$$Mg\sin\theta = \left(\frac{I}{R^2} + M\right)a$$

$$\Rightarrow a = \frac{1}{1 + I/(MR^2)}\, g\sin\theta$$

(2) energy conservation

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2$$

$$\Rightarrow g\frac{dh}{dt} = v\frac{dv}{dt}\left(1 + I/(MR^2)\right)$$

$$\frac{dh}{dt} = v\sin\theta, \quad a = \frac{dv}{dt} \Rightarrow a = \frac{g\sin\theta}{1 + I/(MR^2)}$$

comment: (1) $f < Mg \cos\theta \cdot \mu$, hence, μ cannot be too small to achieve roll without slipping.

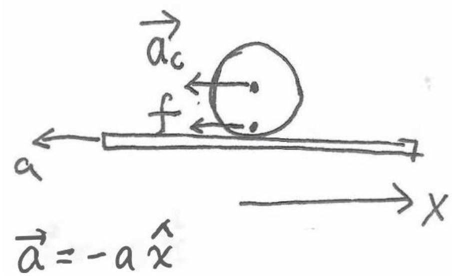
(2) the largest angle for rolling without slipping

$$g\sin\theta\left(1 - \frac{1}{1 + I/(MR^2)}\right) \leqslant g\cos\theta \cdot \mu$$

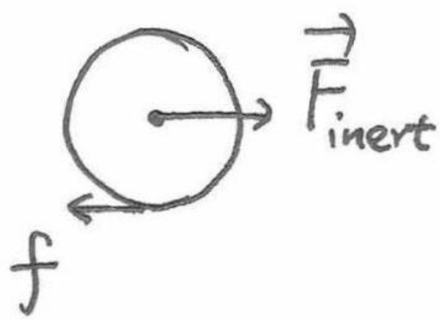
$$\Rightarrow \tan\theta \leqslant \frac{\mu\left(1 + I/(MR^2)\right)}{I/(MR^2)} = \mu\left(\frac{MR^2}{I} + 1\right)$$

Example: A cylinder on a rug

Pull the rug with an acceleration $\vec{a}$, and the cylinder does not slip.



Then describe the motion of the cylinder.
We use the non-inertial frame of the rug.



Then in this frame, there's the inertial force

$$\vec{F}_{\text{inert}} = Ma\hat{x}$$

The torque:

$$\frac{d\omega}{dt} = \frac{a'}{R} = \frac{f \cdot R}{I}$$

and

$$\vec{F} + \vec{f} = M\vec{a}'$$

a' is the acceleration of the cylinder center of mass in the rug frame.

$$\Rightarrow \quad Ma' = Ma - \frac{Ia'}{R^2}$$

$$\Rightarrow \quad a' = \frac{M}{M + I/R^2}a = \frac{1}{1 + \frac{I}{MR^2}}a = \frac{2}{3}a \quad \text{if } I = \frac{1}{2}MR^2$$

The rug acceleration $\vec{a} = -a\hat{x} \Rightarrow$ the cylinder

$$\vec{a}_c = \vec{a} + \vec{a}' = -\frac{I/(MR^2)}{1 + \frac{I}{MR^2}} a\hat{x} = -\frac{I}{MR^2 + I} a\hat{x}$$
$$\Rightarrow \vec{a}_c = -\frac{a}{3}\hat{x}, \quad \text{and } f = Ma'\hat{x} - Ma\hat{x} = -\frac{M}{3} a\hat{x}$$