Lecture 12 Appendix: Conic Sections (圆锥曲线)

The **conic sections**（圆锥曲线）are the family of curves obtained by intersecting a plane with a double-napped right circular cone（双圆锥面）. Their geometric shapes depend solely on the angle between the cutting plane and the cone’s axis（轴线）.

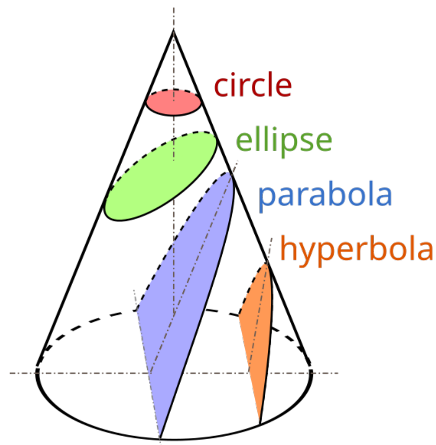

Figure 1: Conic sections formed by slicing a double-napped cone at different angles（圆锥体被平面斜切的几何示意）.

§ 1. Definition（定义）

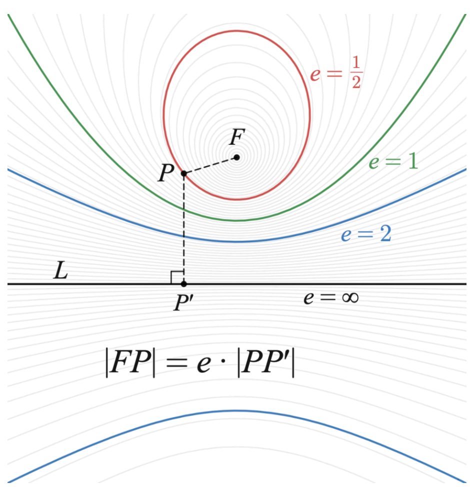

Figure 2: Conic sections general definition.

A conic section is the **locus of all points**（点的轨迹）P such that the ratio of its distance to a fixed point F (called the **focus 焦点**) and to a fixed line L

(called the **directrix 准线**) is constant:

$$\frac{FP}{PP'} = e.$$

The constant e is called the **eccentricity（离心率）**.

- $e < 1$: **ellipse（椭圆）**;
- $e = 1$: **parabola（抛物线）**;
- $e > 1$: **hyperbola（双曲线）**.

In Cartesian coordinates（直角坐标系）the general quadratic equation of a conic is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, C, D, E, F are constants. The discriminant $\Delta = B^2 - 4AC$ determines the type of conic:

$$\begin{cases} \Delta < 0 & \text{ellipse or circle（椭圆或圆）}, \\ \Delta = 0 & \text{parabola（抛物线）}, \\ \Delta > 0 & \text{hyperbola（双曲线）}. \end{cases}$$

§ 2. Polar Equation（极坐标方程）

When the focus is placed at the origin（原点）and the directrix is perpendicular to the polar axis（极轴）, the conic can be expressed in polar coordinates（极坐标系）as:

$$r = \frac{c}{1 + e\cos\theta},$$

where c is called the **semi-latus rectum（半通径）**, a constant determining the scale of the curve.

- Ellipse ($e < 1$): $r = \dfrac{c}{1 + e\cos\theta}$;
- Parabola ($e = 1$): $r = \dfrac{c}{1 + \cos\theta}$;
- Hyperbola ($e > 1$): $r = \dfrac{c}{1 + e\cos\theta}$.

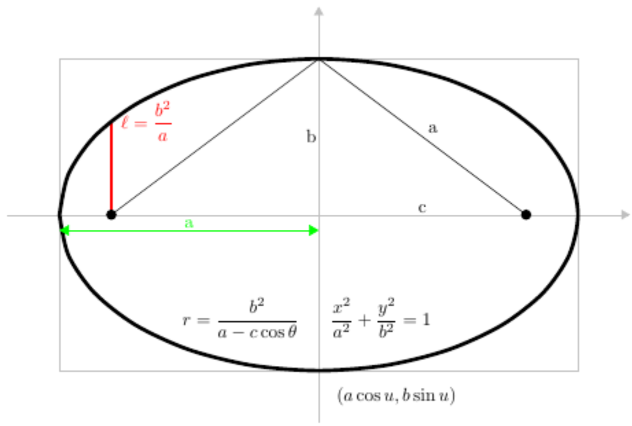

Figure 3: Conic sections in polar coordinates（极坐标下的圆锥曲线示意）.

§ 3. Generation by Plane–Cone Intersection（由圆锥体切割生成）

Consider a right circular cone（直圆锥）with half-angle α. When a plane cuts the cone at an angle β relative to its axis:

- $\beta > \alpha$: ellipse（椭圆）;
- $\beta = \alpha$: parabola（抛物线）;
- $\beta < \alpha$: hyperbola（双曲线）.

Thus all conic sections can be viewed as plane–cone intersections, and their eccentricity depends on these angles:

$$e = \frac{\cos\beta}{\cos\alpha}.$$