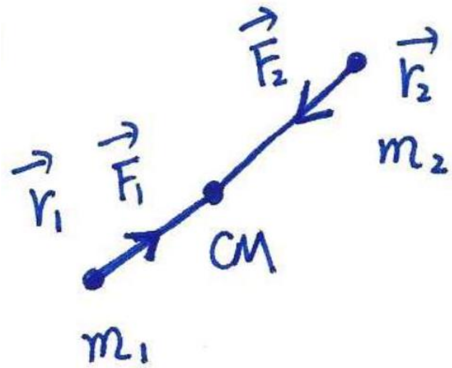
Lect 12: Gravity & Plenary Motion



- CM and relative coordinates: reduced mass

$$\begin{cases} \vec{F}_1\big(|\vec{r}_1-\vec{r}_2|\big) = -\vec{F}_2\big(|\vec{r}_1-\vec{r}_2|\big) \\ m_1\ddot{\vec{r}}_1 = \vec{F}_1 \\ m_2\ddot{\vec{r}}_2 = \vec{F}_2 \end{cases} \tag{1}$$

$\vec{F}_1 + \vec{F}_2 = 0 \Rightarrow \ddot{\vec{R}} = 0$ with $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$ $\leftarrow$ *center of mass coordinate*

$\dfrac{\vec{F}_1}{m_1} - \dfrac{\vec{F}_2}{m_2} \Rightarrow \ddot{\vec{r}} = \left(\dfrac{1}{m_1} + \dfrac{1}{m_2}\right)\vec{F}_1$,where $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the *relative coordinate.*

$$\mu = \frac{m_1 m_2}{m_1+m_2} \ \leftarrow \ \text{reduced mass, } \ \mu < m_1, m_2.$$

- $\mu\ddot{\vec{r}} = \vec{F}_1(|r|)$, separation of center-of-mass motion and relative motion.
- For the relative motion, it is reduced to a single mass point moving in a central force field $\vec{F}_1(|r|)$, with mass replaced by μ.
- $T = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2$

$$= \frac{1}{2}m_1\left[\dot{\vec{R}}^2 + \left(\frac{m_2}{M}\right)^2\dot{\vec{r}}^2 + 2\,\dot{\vec{R}}\cdot\dot{\vec{r}}\,\frac{m_2}{M}\right] + \frac{1}{2}m_2\left[\dot{\vec{R}}^2 + \left(\frac{m_1}{M}\right)^2\dot{\vec{r}}^2 - 2\,\dot{\vec{R}}\cdot\dot{\vec{r}}\,\frac{m_1}{M}\right]$$
$$= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2,$$

with $M = m_1 + m_2$ and $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$.

- $E = T + U = \dfrac{1}{2}M\dot{\vec{R}}^2 + \underbrace{\dfrac{1}{2}\mu\dot{\vec{r}}^2 + U(r)}_{\text{relative motion}}$

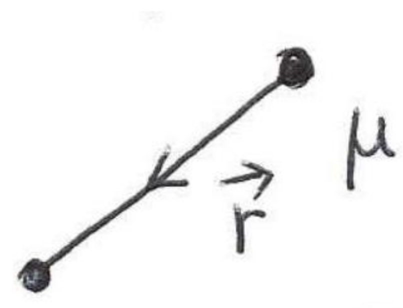
- $\vec{L}_{\rm CM}$ in the CM frame, i.e., the frame in which $\vec{R}$ is at rest.

$$\begin{aligned}
\vec{L}_{\rm CM} &= \left(\vec{r}_1 - \vec{R}\right) \times m_1 \left(\dot{\vec{r}}_1 - \dot{\vec{R}}\right) + \left(\vec{r}_2 - \vec{R}\right) \times m_2 \left(\dot{\vec{r}}_2 - \dot{\vec{R}}\right) \\
&= \frac{m_2}{M}\vec{r} \times m_1 \frac{m_2}{M}\dot{\vec{r}} + \left(-\frac{m_1}{M}\vec{r}\right) \times m_2 \left(-\frac{m_1}{M}\right)\dot{\vec{r}} \\
&= \frac{m_1 m_2}{M}\left(\frac{m_1+m_2}{M}\right)\vec{r} \times \dot{\vec{r}} = \mu\,\vec{r} \times \dot{\vec{r}}.
\end{aligned}$$

- Reduction to 1D motion

We have reduced the two-body problem into a single-body problem in 3D. Now further reduce it to 2D and to 1D motion. In the CM frame, $\vec{L}_{\rm CM}$ is conserved.

The force passes the origin → no torque.
(Angular momentum conservation due to spatial isotropy).

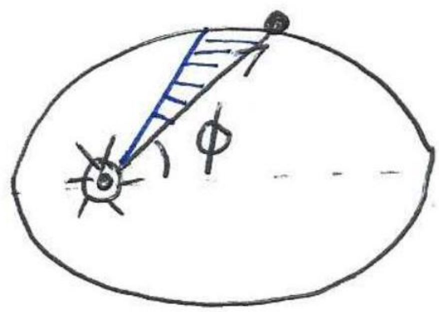


$$\frac{d}{dt}\vec{L}_{\rm CM} = 0 \Rightarrow \vec{L}_{\rm CM} \equiv \text{const vector}$$

$\vec{L}_{\rm CM}$ is perpendicular to the orbital plane ⇒ the motion is coplanar,
e.g., in the xy-plane, and $\vec{L}_{\rm CM} = l\,\hat{z}$.

Then use the equation of motion in polar coordinates:

$$\begin{cases} F_r = \mu\left(\ddot{r} - r\dot{\phi}^2\right) \\ F_\phi = \mu(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = \dfrac{1}{r}\,\mu\,\dfrac{d}{dt}\left(r^2\dot{\phi}\right) \end{cases} \tag{2}$$



$F_\phi = 0 \Rightarrow \dfrac{d}{dt}\Big[\mu r^2 \dot{\phi}\Big] = 0 \leftarrow$ This is Kepler's 2nd law.
Actually

$$\vec{L}_{\rm CM} = l\,\hat{z} = \mu r\hat{r} \times \vec{v} = \mu r\hat{r} \times \left[\dot{r}\,\hat{r} + r\frac{d\hat{r}}{d\phi}\dot{\phi}\right] = \mu r^2\dot{\phi}\,[\hat{r}\times\hat{\phi}] = \mu r^2\dot{\phi}\,\hat{z}$$

$$\Rightarrow \mu r^2\dot{\phi} = l \Rightarrow \dot{\phi} = \frac{l}{\mu r^2} \Rightarrow r\dot{\phi}^2 = \frac{l^2}{\mu^2 r^3}$$

$$\Rightarrow\ F_r = \mu\ddot{r} - \frac{l^2}{\mu r^3} \Rightarrow \mu\ddot{r} = F_r + \frac{l^2}{\mu r^3} \rightarrow \text{Effective 1D motion}$$

Similarly, we can apply previous knowledge of 1D motion and derive:

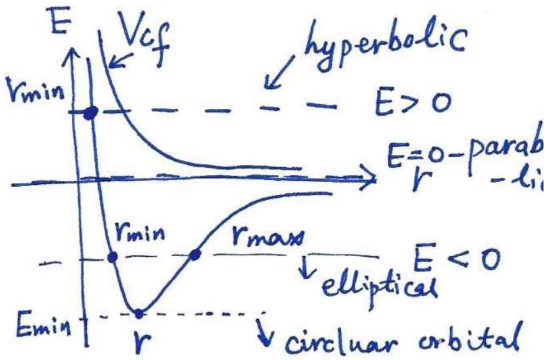
$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{U(r) + \frac{l^2}{2\mu r^2}}_{U_{\rm eff}(r)}, \qquad U(r) = -\int_{r_0}^{r} F_r\,dr.$$

The effect of angular momentum is included by $\dfrac{l^2}{2\mu r^2} \equiv V_{\rm cf}(r)$.

For Kepler problem

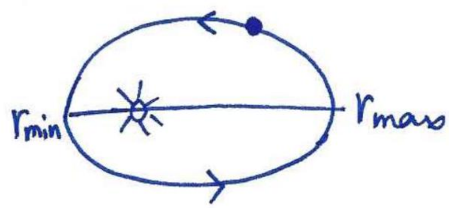
$U(r) = -\dfrac{Gm_1m_2}{r} = -\dfrac{\gamma}{r}$ with $\gamma = Gm_1m_2$.
$U_{\rm eff}(r) = -\frac{\gamma}{r} + \frac{l^2}{2\mu r^2}$

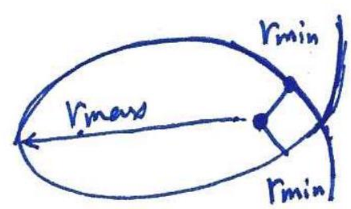
(1) $E < 0$: bound orbital; at $E_{\rm min}$ the radial motion is at rest $\rightarrow$ circular motion.
(2) $E = 0$: unbounded orbitals, parabolic.
(3) $E > 0$: unbounded orbitals, hyperbolic.

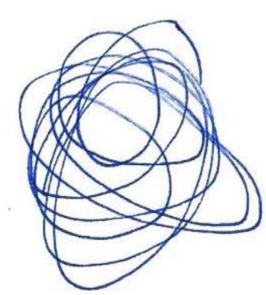


What's special about $1/r^2$ *force field?* — **closed orbit**!

(1) The period of radial motion (bounce) is the same as the angular period $\phi : 0 \sim 360°$.







(2) For a general central force, the orbit may **not be closed**! The ellipse may precess. The angular period isn't the same as the radial period.

Solve the equation of orbit

$$\begin{cases} \mu\ddot{r} = F_r + \dfrac{l^2}{\mu r^3} \\ \dot{\phi} = \dfrac{l}{\mu r^2} \end{cases} \tag{3}$$

Solve $r(\phi)$. Define $u = 1/r$ and replace $\dfrac{d}{dt}$ by $\dot{\phi}\dfrac{d}{d\phi}$:

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{l}{\mu r^2}\frac{d}{d\phi} = \frac{lu^2}{\mu}\frac{d}{d\phi}, \quad \dot{r} = -\frac{l}{\mu}\frac{du}{d\phi}, \quad \ddot{r} = -\frac{l^2u^2}{\mu^2}\frac{d^2u}{d\phi^2}.$$

Thus

$$-\frac{l^2u^2}{\mu^2}\frac{d^2u}{d\phi^2} = \frac{1}{\mu}F_r + \frac{l^2}{\mu^2}u^3 \;\Rightarrow\; \frac{d^2u}{d\phi^2} = -u(\phi) - \frac{\mu}{l^2u^2}F_r.$$

Plug in $F_r = -\dfrac{\gamma}{r^2} = -\gamma u^2$:

$\dfrac{d^2u}{d\phi^2} = -u + \dfrac{\mu\gamma}{l^2}$ $\leftarrow$ inhomogeneous 2nd-order linear differential equation.

Solution: $u = A\cos(\phi - \delta) + \dfrac{\mu\gamma}{l^2}$. Choosing the x-axis along δ (major axis),

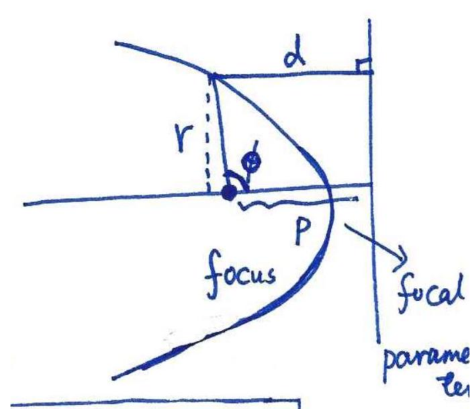
$$\frac{1}{r} = \frac{\mu\gamma}{l^2}\,[1 + e\cos\phi], \quad e = \frac{Al^2}{\mu\gamma},$$

so

$$r(\phi) = \frac{c}{1 + e\cos\phi}, \qquad c = \frac{l^2}{\mu\gamma}.$$

§ Conic curves/sections

p: focal parameter; e: eccentricity.

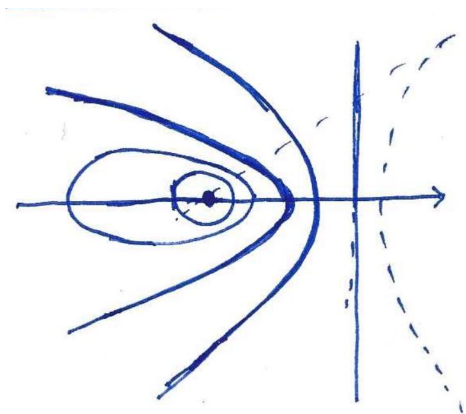


$$e = \frac{r}{d},\; d = p - r\cos\phi \;\Rightarrow\; ed = ep - er\cos\phi = r$$

$$\Rightarrow \boxed{r = \tfrac{ep}{1+e\cos\phi}}.$$

$$\boxed{0 < e < 1 \text{ — ellipse;} \quad e = 1 \text{ — parabola;} \quad e > 1 \text{ — hyperbola.}}$$

Change to Cartesian coordinates:



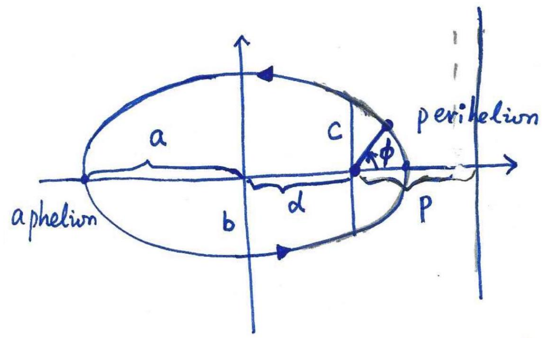
$$r = ep - er\cos\phi \text{ (since } r\cos\phi = x), \quad x^2 + y^2 = (ep)^2 + e^2x^2 - 2e^2px,$$

$$(1-e^2)\left[x + \frac{e^2p}{1-e^2}\right]^2 + y^2 = \frac{e^2p^2}{1-e^2}.$$

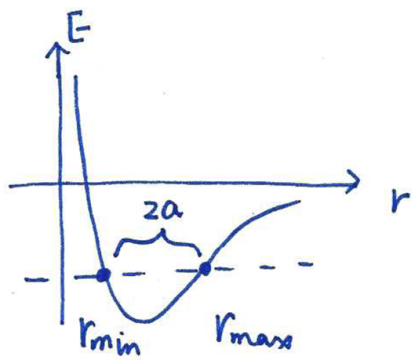
For $0 < e < 1$,

$$\frac{\left(x + \frac{e^2p}{1-e^2}\right)^2}{\left(\frac{ep}{1-e^2}\right)^2} + \frac{y^2}{\left(\frac{ep}{\sqrt{1-e^2}}\right)^2} = 1$$

$$\Rightarrow \begin{cases} a = \dfrac{c}{1-e^2}, \\ b = \dfrac{c}{\sqrt{1-e^2}}, \end{cases} \qquad \begin{cases} c = ep = \dfrac{l^2}{\mu\gamma}, \\ d = \dfrac{e^2p}{1-e^2} = ea, \end{cases} \qquad \begin{cases} e = \dfrac{Al^2}{\mu\gamma}, \\ p = \dfrac{1}{A}. \end{cases}$$



Express the orbit using conserved quantities.



- Energy: using the effective potential

$$U_{\text{eff}}(r) = -\frac{\gamma}{r} + \frac{l^2}{2\mu r^2}, \qquad r_{\min} = \frac{c}{1+e} = \frac{l^2}{\mu\gamma(1+e)}.$$

At $r_{\min}$,

$$E = -\frac{\gamma}{r_{\min}} + \frac{l^2}{2\mu r_{\min}^2} = \frac{\gamma^2\mu}{2l^2}(e^2-1) = \frac{-\gamma}{2a},$$

- The semi-major axis $a = \frac{\gamma}{-2E}$ is determined only by the **energy**.
- The semi-latus rectum ("cord length") $c = \dfrac{l^2}{\mu\gamma}$ is determined only by the **angular momentum**.
- $a = \dfrac{c}{1-e^2} \Rightarrow 1-e^2 = \dfrac{c}{a} = \dfrac{l^2}{\mu\gamma}\cdot\dfrac{-2E}{\gamma} \Rightarrow e = \sqrt{1+\dfrac{2l^2E}{\mu\gamma^2}}.$

$$\frac{b^2}{a^2} = 1-e^2 \Rightarrow \frac{b^2}{a} = (1-e^2)a = c \Rightarrow b = \sqrt{\frac{l^2}{-2\mu E}}.$$

Kepler's 3rd law

$$d\vec{A} = \frac{1}{2}\vec{r} \times d\vec{r} \Rightarrow \frac{dA}{dt} = \frac{1}{2}\frac{l}{\mu}.$$

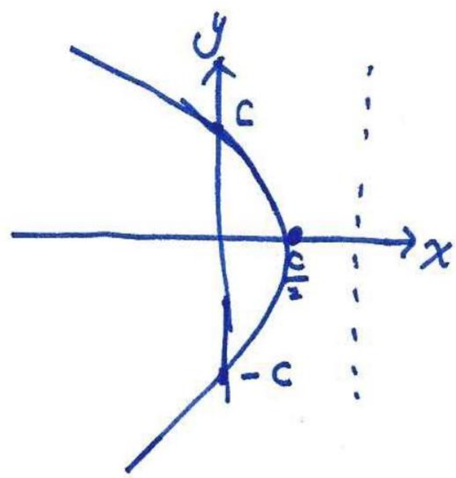
Total area $A = \pi ab \Rightarrow \tau = \frac{A}{dA/dt} = \frac{2\pi ab\,\mu}{l}$, hence

$$\tau^2 = \frac{4\pi^2 a^2 a^2 (1-e^2)\mu^2}{l^2} = \frac{4\pi^2 a^3 c\,\mu^2}{l^2} = \frac{4\pi^2 a^3 \mu}{\gamma} \quad (\text{use } c = \frac{l^2}{\mu\gamma}),$$

$$\Rightarrow \frac{\tau^2}{a^3} = \frac{4\pi^2 \mu}{\gamma} = \frac{4\pi^2}{Gm_{\text{sun}}}, \quad \gamma = Gm_1 m_2 = G\mu(m_{\text{sun}} + m_{\text{earth}}) \approx G\mu m_{\text{sun}}.$$

Orbits:

Unbounded orbits: $r(\phi) = \frac{c}{1 + e\cos\phi}$.

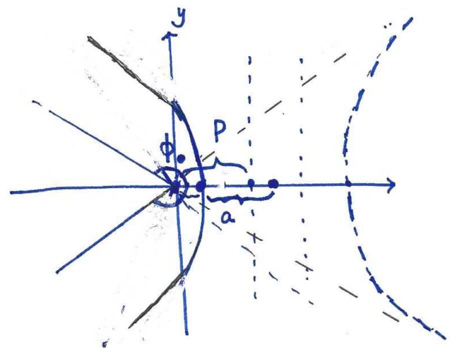


(1) **e=1**: $\Rightarrow r(\phi = \pi) \to +\infty, \quad y^2 = -2c\left[x - \frac{c}{2}\right].$

(2) $\mathbf{e > 1}$:

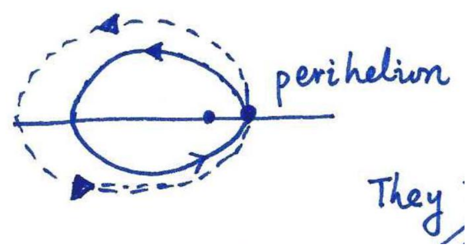
$$\frac{\left(x - \frac{ec}{e^2-1}\right)^2}{\left(\frac{c}{e^2-1}\right)^2} - \frac{y^2}{\left(\frac{c}{\sqrt{e^2-1}}\right)^2} = 1,$$

$$p = \frac{c}{e}, \quad \text{perihelion}\, \frac{c}{1+e}, \quad a = \frac{c}{e^2-1}, \text{center} \left[\frac{ec}{e^2-1}, 0\right].$$



Define $\phi_0 = \cos^{-1}(1/e)$, r is finite when $-\phi' < \phi < \phi'$, ($\phi' = \pi - \cos^{-1}(1/e)$).

Changing orbit



Change from an elliptic orbit with (c_1, e_1) to another with (c_2, e_2)(tangent at perigee):

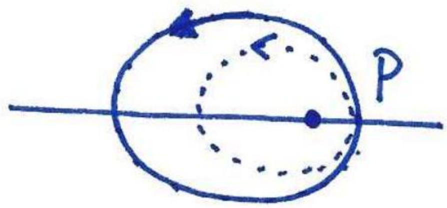
$$\frac{c_1}{1+e_1} = \frac{c_2}{1+e_2}, \qquad l_2 = \lambda\, l_1.$$

Since $c = \dfrac{l^2}{\mu\gamma}$, $\Rightarrow$ $c_2 = \lambda^2 c_1$,

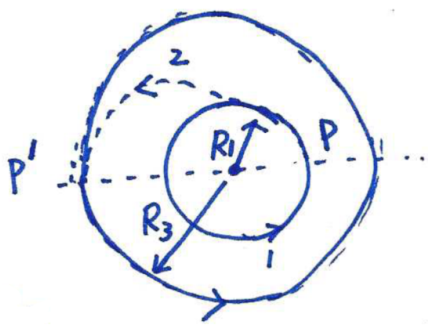
$$\Rightarrow \frac{1+e_2}{1+e_1} = \frac{c_2}{c_1} = \lambda^2 \quad \Rightarrow \quad e_2 = \lambda^2 e_1 + (\lambda^2 - 1).$$

1. If $\lambda > 1$, then $e_2 > e_1$; same perigee, orbit becomes larger and more elliptical. At $e_2 \geq 1$, the orbit becomes open $\to$ parabola/hyperbola.

2. If $\lambda < 1$, then $e_2 < e_1$; the new orbit becomes smaller and less elliptical. At $e_2 = 0$, the orbit becomes circular. How about $e_2 < 0$? then the equation of orbit changes to $r(\phi) = \dfrac{1}{1 - e_2 \cos\phi}$ the perigee and apogee switch.



Changing between circular orbits.



Orbit 1: $e_1 = 0$, $c_1 = R_1$. Orbit 2: eccentricity e_2.

$$\begin{cases} r = \dfrac{c_2}{1+e_2\cos\phi}, \Rightarrow \; \dfrac{c_2}{1+e_2} = \dfrac{\lambda^2 R_1}{1+e_2} = R_1 \Rightarrow e_2 = \lambda^2 - 1, \\ c_2 = \lambda^2 R_1. \end{cases}$$

Apogee:

$$\frac{c_2}{1-e_2} = R_3 \;\Rightarrow\; c_2 = R_3(1-e_2) \;\Rightarrow\; \lambda^2 R_1 = R_3(2-\lambda^2) \;\Rightarrow\; \lambda^2 = \frac{2R_3}{R_1+R_3}.$$

Second thrust: $\; r = C_3 = R_3,\; e_3 = 0,\; C_3 = \lambda'^2 C_2$

$$\Rightarrow\; \lambda'^2 = \frac{C_3}{C_2} = \frac{R_3}{\lambda^2 R_1} = \frac{R_1+R_3}{2R_1}.$$

Final and initial speeds:

$$\begin{cases} v_3 = v_{2,\mathrm{app}}\lambda', \\ \lambda v_1 = v_{2,\mathrm{peri}}, \end{cases} \quad \textit{and} \quad v_{2,\mathrm{app}} R_3 = v_{2,\mathrm{peri}} R_1,$$

$$\Rightarrow\; v_3 = \lambda'\lambda\,\frac{R_1}{R_3}\,v_1 = \sqrt{\frac{R_1}{R_3}}\,v_1.$$

Cosmic velocities

Newton's solution to Kepler's problem paved the way for the space age, starting from the launch of Sputnik 1 in 1957. Below we explain the calculation of the three cosmic velocities. The first astronaut was Yuri Gagarin (1934–1968).

1st cosmic velocity — the orbiting velocity

The first cosmic velocity means that an object does not fall on the ground but orbits around the Earth.

$$m\frac{v_1^2}{R} = \frac{GMm}{R^2}$$
$$v_1^2 = \frac{GM}{R}$$

Since $g = GM/R^2$,

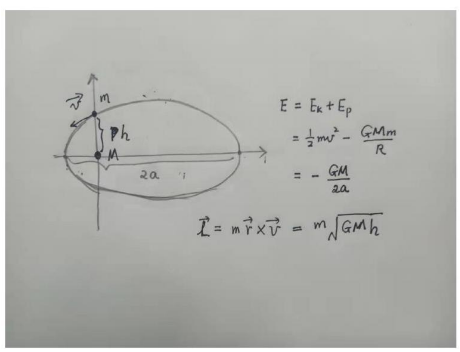
$$v_1 = \sqrt{Rg}.$$

The period T is

$$T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{R}{g}}.$$

Plugging in $R = 6400\,\mathrm{km}$ and $g \approx 10\,\mathrm{m/s^2}$,

$$v_1 \approx 8\,\mathrm{km/s}, \quad T \approx 5024\,\mathrm{s} \approx 84\,\mathrm{min}.$$



The total energy of an elliptic orbit is determined by the semi-major axis a as $E = -\frac{GMm}{2a}$. The angular momentum is determined by the semi-latus rectum h as $l = m\sqrt{GMh}$.

Escaping velocities

The total energy:

$$E = E_k + E_p = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}.$$

The angular momentum:

$$\vec{L} = l\,\hat{z} = m\vec{r} \times \vec{v} = m\sqrt{GMh} = mh\sqrt{\frac{GM}{h}}.$$

For all orbits with the same energy E, they share the same semi-major axis a, but the orbital angular momentum differs; the circular orbit has the largest l. For all orbits with the same h, they share the same l, but their energies are different. Since $l = mrv\sin\theta$, taking $\theta = \pi/2$ minimizes v for fixed l, leading to minimal energy.

2nd cosmic velocity

The second cosmic velocity v_2 is the minimal velocity for escaping from Earth:

$$\frac{1}{2}mv_2^2 - \frac{GMm}{R} = 0 \;\Rightarrow\; v_2 = \sqrt{\frac{2GM}{R}} = \sqrt{2}\,v_1 \approx 11.2\,\mathrm{km/s}.$$

At $v = \sqrt{2GM/R}$ the orbit is parabolic with $E = 0$; for larger v it is hyperbolic with $E > 0$.

3rd cosmic velocity

The third cosmic velocity v_3 is the minimal launch speed (from Earth) to escape the solar system. Earth's orbital speed (Earth–Sun distance $R_e = 1.5 \times 10^8\,\mathrm{km}$, period 1 year):

$$v_0 = \frac{2\pi R_e}{T} \approx 30\,\mathrm{km/s}.$$

Solar escape speed at Earth's orbit:

$$v_{\mathrm{es}} = \sqrt{2}\,v_0 \approx 42.4\,\mathrm{km/s}.$$

Nevertheless, the 3rd cosmic velocity is the object velocity when launched with respect to the earth surface, which can take the advantage of the earth orbiting velocity. Let us consider three steps of launching a rocket to fly away from the earth. During these steps, the distance of the rocket with respect to the Sun changes very little, hence, its potential energy due to the gravity from the Sun can be approximately as a constant. We only count the kinetic energies of the rocket, the earth, the rocket-earth potential energy, and the chemical energy of the fuel. The first stage is before the launch. The earth and the rocket have the same velocity v_0, and the energy stored in the chemical fuel E_c. The total energy is:

$$E_1 = \frac{1}{2}(m+M)v_0^2 + E_{ch} - \frac{GMm}{R},$$

The 2nd stage is that the rocket just acquires the 3rd cosmic velocity v_3 by burning out the chemical fuel, but is still very close to the Earth's surface. Then

$$E_2 = \frac{m}{2}(v_0+v_3)^2 + \frac{M}{2}(v_0+\Delta v)^2 - \frac{GMm}{R},$$

where Δv is the recoil of the earth. According to momentum conservation, we have

$$(m+M)v_0 = m(v_0+v_3) + M(v_0+\Delta v) \Rightarrow mv_3 + M\Delta v = 0.$$

Hence

$$E_2 = \frac{1}{2}(m+M)v_0^2 + \frac{m}{2}v_3^2 + \frac{M}{2}\Delta v^2 - \frac{GMm}{R}.$$

Energy conservation $E_1 = E_2$ gives

$$E_{ch} = \frac{m}{2}\left(1+\frac{m}{M}\right)v_3^2 \approx \frac{m}{2}v_3^2.$$

Final stage (leaving with $\sqrt{2}v_0$ relative to Sun):

$$E_3 = \frac{1}{2}m(\sqrt{2}v_0)^2 + \frac{M}{2}(v_0+\Delta v')^2,$$

where $\Delta v'$ is the recoil of the earth at the end of the 3rd stage. According to the momentum conservation,

$$(m+M)v_0 = m\sqrt{2}\,v_0 + M(v_0+\Delta v') \Rightarrow M\Delta v' = -m(\sqrt{2}-1)v_0.$$

Thus

$$E_3 = \frac{m}{2}(\sqrt{2}v_0)^2 + \frac{M}{2}v_0^2 - m(\sqrt{2}-1)v_0^2.$$

Energy conservation $E_1 = E_3$ yields

$$v_3^2 = v_0^2(\sqrt{2}-1)^2 + v_2^2.$$

With $v_0 = 30\,\mathrm{km/s}$ and $v_2 = 11.2\,\mathrm{km/s}$,

$$v_3 = \sqrt{30^2 \times 0.414^2 + 11.2^2} = 16.7\,\mathrm{km/s}.$$