Lecture 11: Angular momentum conservation

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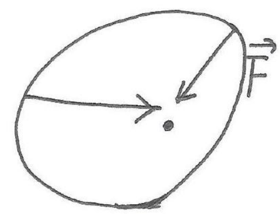
Recap: conservation laws from symmetry (elementary)

- Energy conservation $\leftrightarrow$ time-translation symmetry.
- Linear momentum conservation $\leftrightarrow$ space-translation symmetry.
  Every point in space is equivalent; an isolated body retains its velocity.
- Angular momentum conservation $\leftrightarrow$ *spatial isotropy* (rotational symmetry).

§ Central force field



$$\vec{F}(\vec{r}) = f(r)\,\hat{e}_r$$

Define $\vec{L} = \vec{r} \times \vec{p}$

$$\frac{d\vec{L}}{dt} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F} = \vec{r} \times \vec{F}$$

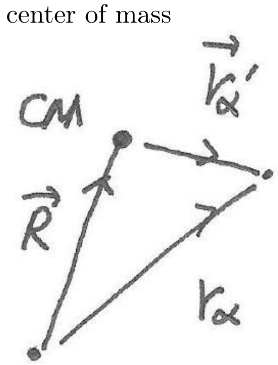
$\vec{r} \times \vec{F}$ is defined as **torque**, which vanishes for a central force field. We know that Kepler's 2nd law is due to angular momentum conservation.

$$\frac{d\vec{L}}{dt} = 0 \leftarrow \text{ central force field.}$$

§ Angular momentum for several particles

$$\alpha = 1, 2, \cdots, N$$

$\vec{R}$: center of mass coordinates
$\vec{r}_{\alpha'}$: vector in the frame of center of mass



$$\vec{l}_\alpha = \vec{r}_\alpha \times \vec{P}_\alpha \quad \vec{L}_{\text{tot}} = \sum_{\alpha=1}^{N} \vec{l}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{P}_\alpha$$

$$
\begin{aligned}
\vec{L}_{\text{tot}} &= \sum_{\alpha=1}^{N} \left(\vec{R} + \vec{r}'_\alpha\right) \times \vec{P}_\alpha = \vec{R} \times \sum_{\alpha=1}^{N} \vec{P}_\alpha + \sum_{\alpha=1}^{N} \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}_\alpha \\
&= \vec{R} \times \vec{P}_{\text{tot}} + \sum_{\alpha=1}^{N} \vec{r}'_\alpha \times m_\alpha \left(\dot{\vec{r}}'_\alpha + \dot{\vec{R}}\right) \\
&= \vec{L}_{\text{orb}} + \underbrace{\sum_{\alpha=1}^{N} \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha}_{\text{internal angular momentum}} + \underbrace{\left(\sum_{\alpha=1}^{N} m_\alpha \vec{r}'_\alpha\right)}_{=0} \times \dot{\vec{R}}
\end{aligned}
$$

$$
\vec{L}_{\text{orb}} = \vec{R} \times \vec{P}_{\text{tot}}, \quad \vec{L}_{\text{spin}} = \sum_{\alpha=1}^{N} \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha
$$

$$
\frac{d}{dt}\vec{L}_{\text{orb}} = \underbrace{\dot{\vec{R}} \times \vec{P}_{\text{tot}}}_{=0} + \vec{R} \times \dot{\vec{P}}_{\text{tot}} = \vec{R} \times \vec{F}^{\text{ext}}
$$

$$
\begin{aligned}
\frac{d}{dt}\vec{L}_{\text{tot}} &= \sum_{\alpha=1}^{N} \vec{r}_\alpha \times m_\alpha \ddot{\vec{r}}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \left(\vec{F}_\alpha^{\text{ext}} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta}\right) \\
&= \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{\text{ext}} + \sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta}
\end{aligned}
$$

Due to Newton's 3rd law. $\vec{F}_{\alpha\beta} = -\vec{F}_{\beta\alpha}$

$$
\sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} = \frac{1}{2}\left(\sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}\right) = \frac{1}{2}\sum_{\alpha \neq \beta} \left(\vec{r}_\alpha - \vec{r}_\beta\right) \times \vec{F}_{\alpha\beta}
$$

For central force field,

$$
\vec{r}_\alpha - \vec{r}_\beta \ \parallel \ \vec{F}_{\alpha\beta} \ \Rightarrow \ \sum_{\alpha \neq \beta} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} = 0
$$

Hence

$$
\frac{d}{dt}\left(\vec{L}_{\text{tot}}\right) = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{\text{ext}} = \sum_{\alpha=1}^{N} \vec{r}'_\alpha \times \vec{F}_\alpha^{\text{ext}} + \vec{R} \times \vec{F}^{\text{ext}}
$$

$$
\Rightarrow \ \frac{d}{dt}\left(\vec{L}_{\text{tot}} - \vec{L}_{\text{orb}}\right) = \sum_{\alpha=1}^{N} \vec{r}'_\alpha \times \vec{F}_\alpha^{\text{ext}}
$$

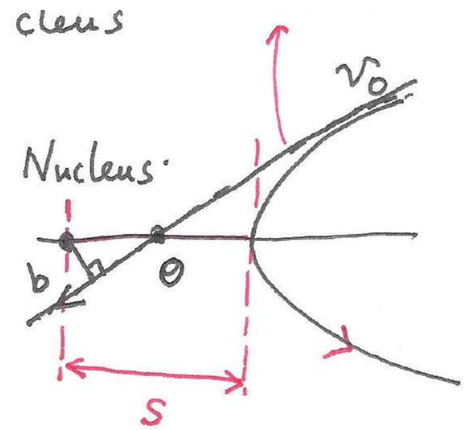
$$
\dot{\vec{L}}_{\text{spin}} = \sum_{\alpha=1}^{N} \vec{r}'_\alpha \times \vec{F}_\alpha^{\text{ext}} \leftarrow \text{torque respect to COM}
$$

In quantum mechanics, angular momentum includes orbital and spin. Spin is quantized at $\hbar/2$. Orbital angular momentum is quantized at $\hbar$.

Example: proton scattering by a heavy nucleus

A proton's trajectory in the repulsive electric force field by a nucleus is a hyperbola. Its asymptote passes the center of the hyperbola, but not the *nucleus*.

Asymptote



This distance from the nucleus to the asymptote "**b**" is called the impact parameter. If we know the velocity v_0 at long distance, i.e. $r \to \infty$, and the impact parameter b, then what's the closest distance that the proton can approach the nucleus?

(1) angular momentum conservation

$$mv_0 b = mv's \;\Rightarrow\; v' = v_0\,\frac{b}{s}$$

(2) energy conservation

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv'^2 + \frac{Qe}{s}$$

$$\Rightarrow \quad \frac{1}{2}mv_0^2 = \frac{1}{2}mv_0^2\left(\frac{b}{s}\right)^2 + \frac{Qe}{s} \quad \text{or} \quad \frac{Qe}{s} = \frac{1}{2}mv_0^2\left[1 - \left(\frac{b}{s}\right)^2\right]$$

Example: shape of galaxy.
Consider a large cluster of dust with spherical shape with an initial angular momentum $\vec{L}$. Then, under gravity, the dust cluster, or, the galaxy begins to collapse. Then the collapses along and perpendicular to the direction of $\vec{L}$ are different.

For a simplified model, consider a mass point m, moving around the center with mass M. The angular momentum L is fixed. Then we can write down an effective potential in the polar coordinate:

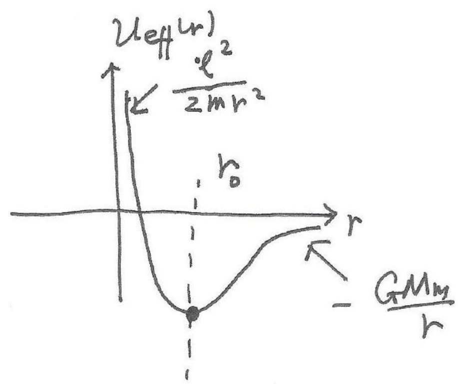
$$
\begin{aligned}
E_k &= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right), \\
&= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
x &= r\cos\theta \Rightarrow \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}, \\
y &= r\sin\theta \Rightarrow \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}
\end{aligned}
$$

$$
E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) - \frac{GMm}{r}
$$

Angular momentum

$$
\begin{aligned}
L = L_z &= m(x\dot{y} - y\dot{x}) \\
&= m\big[r\cos\theta\,(\dot{r}\sin\theta + r\cos\theta\,\dot{\theta}) - r\sin\theta\,(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})\big] \\
&= mr^2\dot{\theta} \\
&\Rightarrow\ \dot{\theta} = l/mr^2
\end{aligned}
$$

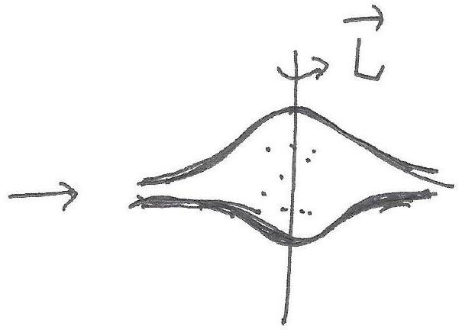


Plug in the expression of E

$$
E = \frac{1}{2} m\dot{r}^2 + \underbrace{\frac{1}{2}\frac{l^2}{mr^2} - \frac{GMm}{r}}_{U_{\text{eff}}},
$$

To minimize $U_{\text{eff}}(r)$:

$$
\begin{aligned}
&\frac{d\,U_{\text{eff}}(r)}{dr} = -\frac{l^2}{mr^3} + \frac{GMm}{r^2}\bigg|_{r=r_0} = 0 \\
&\Rightarrow\quad r_0 = \frac{l^2}{GMm^2}
\end{aligned}
$$



Hence, the collapse in the perpendicular direction is limited to r_0, but the collapse along the direction of $\vec{L}$ has no such constraint.