# Lecture 10: Conservation laws: momentum conservation

- Newton's 3rd law

- Symmetry aspect of momentum conservation

- Collisions

- Tsiolkovsky equation

Newton's 3rd law and its limitations

Action equals reaction. $\vec{F}_{12} = -\vec{F}_{21}$

Problem: Newton's 3rd law assumes instantaneous interactions, but this is *inconsistent with relativity*. The forces between two moving changes in $E\&M$, even the force directions are not parallel to each other.
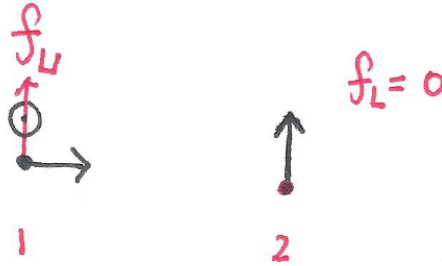
The key of these limitations are solved by the modern perspective: **fields**.

Charge (mass) generates fields. Fields propagate at the light velocity, and fields act on charge (mass). Nevertheless, we still assume that within Newtonian mechanics, *at speeds much less than the speed of light*, Newton's 3rd law is valid.

Consider two objects 1 and 2, $\vec{F}_{12} = -\vec{F}_{21}$.

$$\begin{cases} d\vec{P}_1/dt = \vec{F}_{12} \\ d\vec{P}_2/dt = \vec{F}_{21} \end{cases} \Rightarrow \frac{d\left(\vec{P}_1 + \vec{P}_2\right)}{dt} = 0$$

Symmetry and Galilean transformation (perspective)

Consider a mass of $2m$. Imagine there's an explosion that it splits two parts of equal mass. The system has the reflection symmetry, hence, the velocities are equal in magnitude and opposite in direction.

Hence, the momentum before and after the explosion is zero.

Consider two equal masses with velocities v_1 and v_2 . They collide and stick together, what's their final velocity?

We switch to a reference frame F', which has the velocity $v' = \frac{v_1+v_2}{2}$. In this reference, two masses have the opposite velocities $\pm\frac{v_1-v_2}{2}$, hence. they exhibit the reflection symmetry. Then if they stick together, the velocity has to

be zero. Then transform back to the lab frame $v = \frac{v_1+v_2}{2}$.
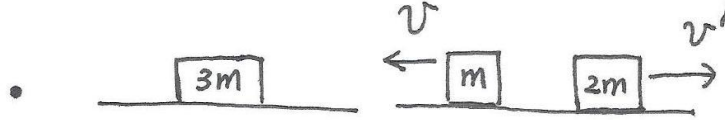
Consider a mass of $3m$. Then it splits into two non-equal masses with m and $2m$, then what's the relation between v and v'?

We view $3m$ first split to masses $2m$ and m. Then $2m$ splits into two equal masses, then

Then based on symmetry considerations, we conclude that if the mass ratio of splitting $m_1 : m_2 = 1 : 2$, then $v_1 : v_2 = 2 : 1$ with opposite directions. Hence, the total momentum remains zero.

If we reverse the process, and perform Galilean transformation, we can arrive that if the mass ratio is $1 : 2$, then after collision, if then stick togetherr we have $3mv = mv_1 + 2mv_2$, i. e. $v = \frac{v_1+2v_2}{3}$.

Using the similar method, we can prove that if a mass split into two parts with the ratio $m_1 : m_2 = p : q$(p and q integers), then their velocities are opposite in direction $v_1 : v_2 = q : p$.
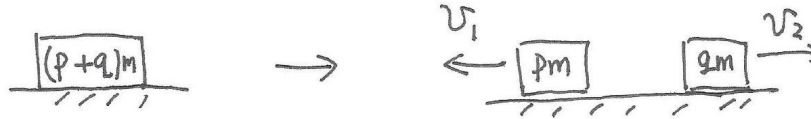
N-interacting point particle systems

$$\begin{cases} m_1\ddot{\vec{r}}_1 = \vec{F}_1^{ex} + \vec{F}_{12} + \vec{F}_{13} + \cdots \vec{F}_{1n} \\ m_2\ddot{\vec{r}}_2 = \vec{F}_2^{ex} + \vec{F}_{21} + \vec{F}_{23} + \cdots \vec{F}_{2n} \\ \vdots \\ m_n\ddot{\vec{r}}_N = \vec{F}_N^{ex} + \vec{F}_{N1} + \cdots + \vec{F}_{w,N-1} \end{cases}$$

$$\Rightarrow \quad \sum m_i\ddot{\vec{r}}_i = \frac{d}{dt}\left(\sum_i m_i\dot{\vec{r}}_i\right) = \sum_{i=1}^{N}\vec{F}_i^{ex} + \sum_{i\neq j}\vec{F}_{ij}$$
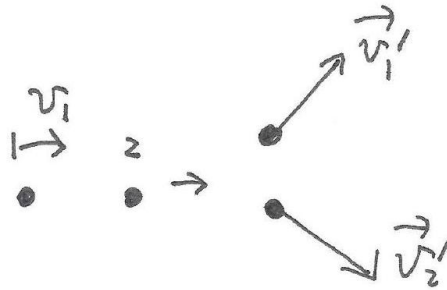
$$\text{Since } \vec{F}_{ij} + \vec{F}_{ji} = 0, \Rightarrow \frac{d}{dt}\left(\sum_{i=1}^{N} m_i\dot{\vec{r}}_i\right) = \vec{F}_{\text{tot}}^{\text{ex}}$$

$$\vec{P}_{\text{tot}} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i = M \frac{d}{dt} \sum_{i=1}^{N} \left(\frac{m_i}{M} r_i \right) \leftarrow M = \sum_{i=1}^{N} m_i$$

$$\vec{P}_{\text{tot}} = M \dot{\vec{R}} \leftarrow \vec{R} = \sum_{i=1}^{N} \frac{m_i}{M} r_i \leftarrow \text{ center of mass position.}$$

This is the origin of the validity of mass point although when applying for an extended object. For translation, an extended object can be viewed as if all the mass is concentrated at its mass center.
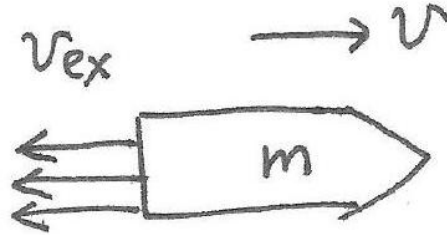
Equal mass, elastic collision

Energy conservation (1) $\frac{1}{2} m v_1^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2$
Momentum conservation (2) $m \vec{v_1} = m \vec{v}_1' + m \vec{v}_2'$

$$\begin{aligned}
(2) &\Rightarrow v_1^2 = v_1^2 + v_2^2 + 2\vec{v}_1' \cdot \vec{v}_2' \\
(1) &\Rightarrow v_1^2 = v_1^2 + v_2'^2
\end{aligned} \qquad \Rightarrow \vec{v}_1' \perp \vec{v}_2'$$

$$\text{or } v_1' \text{ on } v_2' = 0.$$
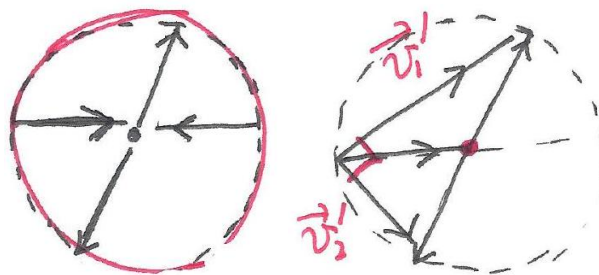
collision in the center of mass frame **v.s.** the velocity in the lab frame

Tsiolkorsky (Rocket equation) 1903

v_{ex}: effective exhaust velocity – the velocity of exhaust gas relative to the rocket.

time t: $P(t) = mv$

time $t + \Delta t$: rocket mass $m + dm$ $(dm < 0)$

fuel burnt: $-dm$

$$p(t + \Delta t) = (m + dm)(v + dv) + (-dm)(v - v_{\text{ex}})$$
$$\approx mv + mdv + v_{\text{ex}}\, dm$$

$$p(t + \Delta t) = p(t) \Rightarrow mdv = -v_{\text{ex}}\, dm$$
$$dv = -v_{\text{ex}}\, \frac{dm}{m} \Rightarrow \int_{v_0}^{v} du = -v_{\text{ex}} \int_{m_0}^{m} \frac{dm}{m}$$
$$v - v_0 = v_{\text{ex}}\, \ln \frac{m_0}{m}$$

if 90% rocket's mass is fuel, $\Rightarrow m_0/m = 10 \Rightarrow$ $\begin{aligned} v - v_0 &= v_{\text{ex}}\, \ln 10 \\ &= 2.3 v_{\text{ex}} \end{aligned}$